A SCHOOL GEOMETRY

ON "NEW SEQUENCE" LINES

BOOKS I–V

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PREFACE

It has been held by some teachers that in the subject of elementary geometry no definite sequence need be universally established. On the other hand, practical difficulties have been found in the absence of a recognised order; and the matter has received definite treatment by the Incorporated Association of Assistant Masters in a Report on this subject, and also by the Scottish Education Department.

To a great extent this text-book of Elementary Geometry has followed the lines laid down in these important Reports.

The authors have striven to make the subject simple as well as systematic. By introducing parallels as well as angles at the beginning it has been possible to lighten some of the work on congruence of triangles, equalities in a triangle, and inequalities of angles and sides.

As in *Elementary Geometry* by the same authors, hypothetical constructions are used in the proofs of Theorems, *e.g.* the existence of the bisector of an angle is assumed before the student has been taught to bisect an angle. This allows of the Theorems and Problems being arranged independently of one another, and gives considerable latitude in the proofs of Theorems.

It is recommended that the Theorems and Problems in each Book should be taken side by side as parallel courses.

The constructions are meant to be accurately drawn with instruments, and the methods employed are practical.

Important riders are printed in prominent type, and in many cases hints for solution are given.

Book II. is concerned with properties of parallelograms, and areas of triangles, quadrilaterals and polygons; and in this, as in other Books, Constructions are given after the Theorems.

Book III. deals with circles.

Book IV. gives further instruction in areas, and applications of the theorem of Pythagoras.

Book V. treats of Proportion. Attention is called to incommensurable quantities; but Euclid's treatment of such quantities is deferred till late in this section.

Books VI. and VII. are devoted to Solid Geometry; but teachers who wish to introduce their pupils to some notions on solid figures at an earlier stage will be able to make use of some parts of these two Books.

There are numerous exercises, including some in Mensuration, interspersed, some easy enough to encourage the learner, some capable of testing ingenuity, and it is hardly too much to hope that some students, after success in these, may attack and solve some subsequent propositions without having seen more than the enunciations.

W. M. B.
A. A. B.

The numbers at the ends of the Exercises refer to their Solutions in the Key to Baker and Bourne's "Elementary Geometry."

CONTENTS

INTRODUCTORY CHAPTER

EXPERIMENTAL GEOMETRY FOR BEGINNERS.

[Much of this chapter may with advantage be read a second time concurrently with later parts of the book.]

Instruments required :

(1) Pair of compasses.
(2) Set-square.
(3) Protractor.
(4) Graduated flat ruler, showing tenths of an inch on one side, centimetres and millimetres on the other.

In working through this chapter, the pupil should not be supplied with any formal definitions. He should learn what things are by seeing them, so that he may, later on, more easily understand and assimilate the formal part of the subject.

First let him learn the meanings of the markings on the protractor and ruler.

EXERCISES A.

1. Take a cube, and see that it has three dimensions—**length, breadth,** and **thickness.**

2. Consider a surface of the cube,—one of its faces—and see that it has only two dimensions,—**length** and **breadth.**

3. Look at an edge of the cube,—a line—and see that it has but one dimension, viz. **length.**

4. How many faces has a cube ?

5. How many lines are formed by the faces of a cube ?

6. We cannot with a pencil make a line—we only represent a line. Why ?

7. What do we have where edges of a cube meet ? A point. What dimensions has it ?

8. We cannot with a pencil make a point—we only represent a point. Why ?

9. Name the number of points that are formed by the intersections of the edges of a cube.

10. What are the boundaries of a solid figure, a cube, or a sphere, or a cone ?

11. What are the boundaries of a surface ?

12. If we join *any* two points in a flat surface—a plane—by a straight line, that straight line lies entirely in the plane. Does this happen if we join *any* two points in a curved surface ? For instance the surface of a sphere, or cone ?

Hence we have a test for a plane.

13. Take one face of the cover of a book to represent a plane. We see that it can be made to turn about the line where it is attached to the back. Use it to show that two intersecting straight lines lie in a plane.

14. How would you test whether a line was straight ?

15. How can you test your ruler to see if it is straight ?

16. How can you test a surface to see if it is flat ?

17. What do you observe about the intersection of two flat surfaces, the intersection of two walls of the room, or the intersection of two faces of a cube ?

18. Fold a piece of paper carefully: the crease forms a straight line. Why is this ? Test the fact.

EXERCISES B.

1. Draw a straight line 3 inches long. Measure it in centimetres, and hence find the number of millimetres in an inch.

2. Repeat the above with a straight line 4 inches long, and see if your answer is the same as before.

3. Draw a straight line 6 centimetres long, measure it in inches and tenths of an inch, and determine the value of a centimetre in inches.

4. Repeat the process with a line 8 centimetres long, and see if your answer comes the same. In each of these two examples you should get your result to the nearest tenth of an inch.

5. Draw a straight line 11·45 centimetres long, and measure it in inches.

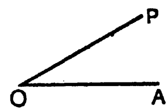
6. Make rectilineal figures (*i.e.* figures contained by straight lines) (1) of three sides (a triangle), (2) of four sides (a quadrilateral), (3) of five sides (a pentagon), (4) of six sides (a hexagon).

7. Make a curvilinear figure having one curve for a boundary.

8. Make a curvilinear figure having more than one curve for its boundaries.

ANGLES.

If two lines AO and PO meet at a point O, an angle is formed. It is called the angle AOP or POA. It does not matter in which of these two orders the letters are used, but the letter O *must* be put between the other two in naming the angle. For abbreviation we may write $\angle$AOP instead of "the angle AOP."



Before explaining what an angle is, we shall find it useful to say what it is not.

It is not an enclosed space; so it cannot be measured in square inches or square feet.

Its size does not depend upon the length of the lines which contain it.

A notion of what an angle is may be got by considering the points of a mariner's compass, and what amount of turning is required to change from facing in one direction to facing in another.

Face East.

Turn until you face North.

Then turn until you face West.

Turn farther to face South, and farther until you face East again.

By a succession of turns you have got back so as to face in the original direction.

The direction in which you originally faced and the direction when you face West are opposite to each other. They are in a straight line.

To measure the angle AOB we have to consider what amount of turning round O is required to move a line from the position OA into the position OB.

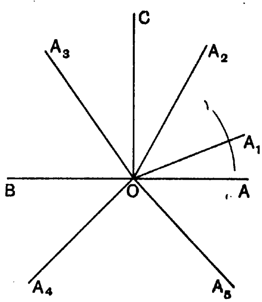
On a flat board rule a straight line AOB. At O attach a fine thread, which may be of any length. It may be attached by fastening it in a small hole made in the board at O.

The thread may be held at first so as to lie along OA. It is then turned round O, being kept tight all the time, till it lies along OA_1. From that position it may be turned till it lies along OA_2, having thus traced out the angles AOA_1, A_1OA_2 in succession.

When half-way between OA and OB it lies along OC and has traced out the angle AOC, which is called a right angle.

Move it further till it lies along OB in a straight line with its first position OA.

When the thread has gone from its first position completely round, so as to be in its first position again, it has made one revolution. The angle which has been traced out is 4 right angles. Half a revolution (or 2 right angles) would bring it into a straight line with its original direction. Thus a revolution means turning through 4 right angles, and a quarter of a revolution is equivalent to turning through one right angle.



The tracing out of an angle may be illustrated on the face of a clock.

In a quarter of an hour the long hand moves through a right angle. In half an hour it moves through 2 right angles, and then occupies a position exactly in a straight line with its position at the beginning of the half-hour.

As the long hand describes a right angle in 15 minutes, the angle described by it in 5 minutes must be $\frac{5}{15}$ or $\frac{1}{3}$ of a right angle.

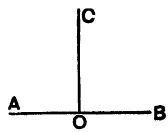
Vertically opposite angles.

If two straight lines AB, CD intersect at E, the angle AEC is said to be vertically opposite to the angle BED, and the angle CEB vertically opposite to the angle DEA.

A straight angle.

When AO and OB form a straight line, the angle AOB is called **a straight angle.**

Let OC be at right angles to AOB. The angle AOB is traced by a line revolving from OA into the position OC and further revolving from OC to OB.



Thus the whole angle AOB is formed by adding the angle COB to the angle AOC.

$\therefore$ the straight angle AOB = angle AOC + angle COB,

i.e. a straight angle = two right angles.

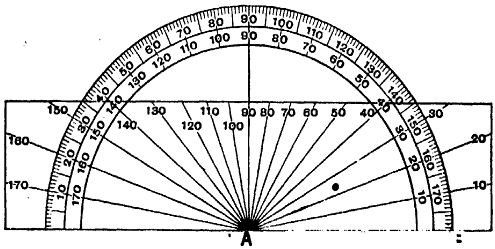
A degree is $\frac{1}{90}$ of a right angle. That is to say if a right angle were divided into 90 equal angles, each of these parts would be a degree.

To denote fractions of a degree we use minutes and seconds.

A right angle contains	90 degrees	(written		90°).
A degree	,,	60 minutes	(,,	60′).
A minute	,,	60 seconds	(,,	60″).

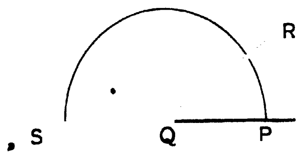
What is the angle between the long hand and the short hand of a clock at 9 o'clock, if we reckon from the long hand in the direction in which movement takes place? *Three* right angles, not one right angle.

DESCRIPTION OF THE PROTRACTOR.



The figure shows two forms of protractor. The object of either is to measure an angle between two given straight lines. To measure an angle PQR place along QP the edge of the protractor so that the central mark A is exactly at Q and the graduation marked 0 is on the line QP.

Observe where the line QR crosses the graduated edge, and read off the number of degrees marked on the protractor at that point. If PQ be produced to S the angle RQS is the supplement of PQR (*i.e.* $\angle RQS + \angle RQP = 180°$); but as one of these angles is less than 90° and the other greater, you will have no difficulty in



reading the one which is required, though the graduations of some protractors are so marked as to show both of these angles. For instance you may find 130° and 50° marked at the same point;

but, if it is an acute angle which you are measuring, you know that the 50° is the result which you require.

In some positions of the angle you may find it more convenient to begin your measurement from the line QR, that is you will put the edge of the protractor along QR and read off the mark where QP crosses the graduated scale.

EXERCISES C.

Measurement of Angles.

Mention the size of the angle between the hands of a clock at the following hours, reckoning from the long hand in the direction of clock-movement:

1. 3 o'clock. **2.** 6 o'clock. **3.** 9 o'clock. **4.** 1 o'clock.

5. 2 o'clock. **6.** 7 o'clock. **7.** 5 o'clock. **8.** 10 o'clock.

9. How many degrees does the long hand of a clock turn through in an hour, 20 minutes, half an hour, 40 minutes?

10. How many degrees does the short hand of a clock turn through in an hour, $\frac{1}{2}$ hour, $\frac{3}{4}$ hour?

11. Measure the angles of your set-square.

12. Draw an angle ABC, and with centre B describe a circle cutting BA and BC at A and C. With centres A and C describe two more circles with equal radii cutting one another at D. Join BD, and measure the angles ABD, CBD. **See that we thus have a geometrical method for bisecting an angle.**

13. With your protractor, make two separate right angles: cut them out, and by placing one upon the other, test their equality.

14. Take a piece of paper with a straight edge. Fold along this edge, *i.e.* so that one part of this edge lies on another part of it, and measure the angles formed by the crease and the edge.

15. Hence find out an experimental method of drawing a right angle.

16. Draw a straight line on a piece of paper, and fold about it. Fold the paper again along this crease, and having opened the paper again, measure the angles between the creases. Hence devise an experimental method for drawing a straight line at right angles to a given straight line at a given point in it.

17. Employ your ruler and set-square to draw a straight line at right angles to a given straight line. Test your result.

18. Bisect a right angle by folding. How do you know that the angle is bisected?

19. Draw any angle on paper, cut it out, and bisect it by folding.

20. Draw an angle of 60°, and an angle of 120°: cut them out, fit them together and see that when thus fitted, two of the bounding edges are in a straight line.

Do the same with angles of 35° and 145°; also with angles of 30° and 150°.

21. Draw two straight lines intersecting one another. Measure two adjacent angles. What do they together come to? Do this experiment with several pairs of straight lines. What do you find out?

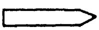
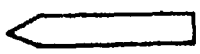
22. Make an angle BAC equal to 60°, and make AB equal to AC. Join BC, and measure the angles at B and C.

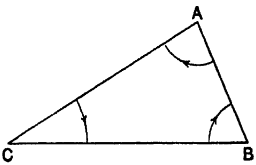
23. The shortest distance between two points is obviously the straight line joining them. Deduce a fact about the sides of a triangle, and verify it by making triangles and measuring.

24. Try to make a triangle whose sides are 4, 5, and 9 centimetres long.

25. Make a triangle and measure all its angles in degrees. Add them together. Do this for several triangles. What do you discover?

Make a triangle, and cut off all its angles. Fit them together, and see that they together make two right angles.

26. Another method. Draw a triangle ABC. Produce AB both ways. Rotate this line about the point A till it lies along AC. Then rotate it about C till it lies along CB, and finally about B till it lies along BA, as shown by the arrows in the figure. See that a pencil started thus , finishes up thus , pointing the other way. The pencil has turned through two right angles, and, in doing so, has turned through the three angles of the triangle. This suggests that the three angles of a triangle are together equal to two right angles.



27. Draw a quadrilateral, and measure all its angles in degrees. Add them together. What do you see?

28. Do the above experiment by cutting off all the angles of a quadrilateral and fitting them together.

29. Draw a straight line AB, and make the angles ABC, BAC each equal to 45°. Measure the angle ACB. What should it be, and why?

30. Take any triangle ABC. Draw the line bisecting the angle A. Draw the perpendicular from A to BC. Join the point A to the middle point of BC. There are three different lines. In what sort of triangle are they all the same?

31. Join all the vertices (angular points) of a triangle to the middle points of the opposite sides. Do they form a triangle? Measure the parts into which they divide each other. What do you find out?

32. Take any triangle ABC. Produce BC to D, and measure the angles ABC, ACD. Which is the greater?

33. Try this with several triangles of different shapes.

34. Make a triangle ABC with AC equal to BC. Produce AC to D, making CD equal to AC. Join BD, and measure the angle ABD.

Construction of triangles with given parts.

(See Book I. Propositions 9-14.)

35. Construct a triangle two of whose sides are 5 cm., 4 cm., the angle between them being 40°.

36. Repeat this, and see whether this second triangle is equal in all respects to the first.

Figures which are equal in all respects are called **congruent.**

37. Construct a triangle with sides 1·4, 1·6, 2·1 inches. Does the order in which you take the sides make any difference?

38. Construct a triangle with a side 2 inches long, and with angles 35° and 55° adjacent to it.

39. Repeat this construction, and convince yourself that the triangle obtained is congruent with the previous one.

Examples of this sort may be used to lead beginners to anticipate the truth of the propositions which prove the congruence of triangles under certain conditions.

EXERCISES D.

1. Placing your ruler on paper, draw lines along its edges. These lines are **parallel**, *i.e.* if you produce them either way in the plane of the paper, they will never meet. Test this.

2. Can two straight lines be such that they will never meet however far we produce them, and yet not be parallel?

Point out two lines in the room which never meet and yet are not parallel.

3. Draw two parallel straight lines with your set-square and ruler.

4. Draw two parallel straight lines, AB, CD. Draw another straight line cutting them at E and F. Measure the angles AEF, BEF, DFE, CFE. Do this in several cases and deduce a general theorem.

Measure the other angles which the straight lines make with one another, and deduce further results.

5. Draw two parallel straight lines, with another straight line EF cutting them. Cut out the figure along the parallel lines and along the line EF. Fit the pieces together, and so prove that if a straight line meets two parallel straight lines, it makes the alternate angles equal.

6. Find other equal angles in the above case.

7. Draw two parallel straight lines, and make them equal in length. Join their ends (not the opposite ends), and see that the joining lines are also equal and parallel.

8. Draw a **rectangle** (a four-sided figure which has all its angles right angles). Cut it out, and also cut it along a diagonal. By superimposing, see that the two triangles so obtained are equal in all respects.
Do the same with a parallelogram (a four-sided figure whose opposite sides are parallel).

9. Show that the diagonal does not bisect the angle unless the sides are all equal.

EXERCISES E.

1. Draw a triangle whose sides are 2, 3, and 4 inches long.

2. Draw a triangle whose sides are 3, 4, and 5 inches long. Measure its greatest angle.

3. Draw a triangle whose sides are 6, 8, and 10 centimetres long. Measure its greatest angle.

4. Draw a triangle whose sides are 4·5, 6, and 7·5 centimetres long. Measure its greatest angle.

5. Draw a triangle whose sides are 1·5, 2, and 2·5 inches long. Measure its greatest angle.

6. From the above four examples, see that a triangle whose sides are 3, 4, and 5 units long (or any multiple of these lengths) is right-angled.

7. Prove the same for triangles whose sides are 5, 12, and 13 units long.

8. Prove the same for triangles whose sides are 7, 24, and 25 units long.

9. On squared paper make a right-angled triangle, taking the sides AB, AC along printed lines. Measure all three sides, and see that $AB^2 + AC^2 = BC^2$. Do this for several triangles.

EXERCISES F.

1. Draw a circle with a radius of 2 inches, and place in it a chord 3 inches long.

2. Describe a circle and place in it chords AB, BC, CD, etc., each equal to the radius. Do this all the way round the circumference, and see what sort of figure you obtain.

3. Draw two diameters of a circle at right angles to one another, and join their extremities. Measure the sides and angles of the quadrilateral you thus obtain.

4. If a straight line cuts a circle at all, it generally cuts it at two points. Take AB a diameter of a circle, and at A draw a straight line at right angles to AB. (This is easily done if you remember that a triangle whose sides are 3, 4, and 5 is right-angled.) What do you see about this line? Such a line is called a **tangent**.

5. With a centre A describe a circle, and take any point B outside it. On AB as diameter, describe another circle cutting the first at D and E. Join BD, BE. What do you see about these lines? Measure them. Join DE, and measure the parts into which it is divided by AB.

6. Draw a straight line AB. With centre A and radius AB describe a circle. With centre B and radius BA describe another circle cutting the first circle at C and D. Join CA, CB. Measure the sides of the triangle ABC. You will find them equal. Give reasons for this. Do the same for the triangle ABD.

7. Do the same as in the above problem, and join CD, cutting AB at E. Measure AE and BE, and see that we thus have **a method for bisecting a given straight line.**

8. Take a straight line AB, and with centres A and B and any radius describe circles cutting one another at C and D. Join CD, and see by measurement that it bisects AB. Will **any** radius do?

9. Draw a straight line 3 inches long, and on it describe an equilateral triangle. See Def., p. 22.

10. Draw a straight line 4 centimetres long, and on it describe an isosceles triangle, whose other sides are 5 centimetres long. See Def., p. 22.

11. Make an isosceles triangle whose vertical angle is 40°.

12. Make several equilateral triangles, and measure all their angles. What do you discover about all equilateral triangles?

13. Make an isosceles triangle and measure the angles at the base. Do this for several triangles. What do you discover?

14. Make an isosceles triangle, and produce the equal sides. Measure the angles on the other side of the base. Do this for several triangles. What do you find out?

15. Make a triangle ABC having AB equal to 3 inches, AC equal to 2 inches, and the angle BAC equal to 45°. Make another triangle in exactly the same way. Cut the triangles out: fit them together, and see that they may be made to coincide, *i.e.* to fit exactly upon one another.

16. Make another pair of triangles in the same way with fresh measurements. Fit them together and deduce a general theorem.

17. Make a pair of triangles with sides of 5, 6, and 7 centimetres. Fit them together, and see that they may be made to coincide.

Do this with other pairs of triangles, and deduce a general theorem.

18. Make a triangle two of whose angles are each equal to 55°. Measure the sides opposite these angles.

Make other triangles each having two equal angles. In each case measure the sides opposite the equal angles, and deduce a general theorem.

19. Make a triangle ABC, and take any point D within it. Join AD, BD. Measure AC, BC, AD, BD, and see that in every case AC + BC is greater than AD + BD.

20. In the above case measure the angles ACB, ADB, and see that in every case the angle ADB is greater than the angle ACB.

21. Make a triangle ABC right-angled at C. Take D the middle point of AB, and join CD. Measure DA, DB, DC. What do you see?

Do the same for several right-angled triangles, and deduce a general theorem.

22. Draw a triangle whose sides are 2, 2·6, and 3 inches long. Take the middle points of its sides, and by cutting along the lines joining these middle points divide the whole triangle into four. Fit these triangles together, and discover something.

See if this is true for all triangles.

EXERCISES G.

1. Describe a circle, and a quadrilateral with its angular points on the circumference. Measure two opposite angles, and add them together. Do this with several circles. What do you discover?

2. Describe a circle, and draw any chord AB. Take C and D any two points on the circumference and on the same side of the line AB. Join CA, CB, DA, DB. Measure the angles ACB, ADB. What do you find? Take another point E, still on the same side of AB, and measure the angle AEB. What theorem do you thus establish?

3. Draw two circles cutting one another, and join their common points. Measure the angle between this line and the line of centres. Also measure the parts into which the common chord is divided by the line of centres.

4. Take C any point in a straight line AB. With centre A and radius AC describe a circle. Also with centre B and radius BC describe another circle. What do you notice about the circles ?

5. Take any point C in a straight line AB. With centre A and radius AB describe a circle, and with centre C and radius CB describe another circle. What do you see about the circles ?

6. What do you find from the two previous exercises ?

7. Draw a circle from a centre O, and take any chord AB, not passing through O. On the circumference take a point C on the same side of AB as the centre. Join OA, OB, CA, CB. Measure the angles AOB, ACB. What do you discover ?

Do a similar experiment with different circles.

8. Describe a circle, and a diameter AB. Take any point C on the circumference, and join CA, CB. Measure the angle ACB. Take another point D on the circumference, and measure the angle ADB. What do you discover ?

EXERCISES H.

Folding.

1. Repeat Exercises C. 14, 15.

2. Repeat Exercises C. 16, 17, 18.

3. Take any piece of paper, and by folding it make a right-angled triangle.

4. With any piece of paper make an isosceles triangle by folding.

5. Draw a straight line AB, and take C any point in it. By folding, find a straight line through the point C at right angles to AB.

6. Draw a straight line AB, and take C any point without it. By folding obtain a straight line through the point C at right angles to AB.

7. Fold a right angle, one leg to the other: open it, and fold the vertex to any marked point in the crease. You can thus get a **square.** Verify this by measurement.

8. Fold similarly any other angle, and get a **rhombus,** or **diamond.** Measure the parts of the diagonals of this rhombus, and the angles between them. What do you discover ?

9. Take any piece of paper and by folding obtain two parallel straight lines.

10. Take any piece of paper and make a **rectangle** by folding. Mark its angular points.

11. Take any piece of paper, and by folding, make a **parallelogram.** Mark its angular points.

EXERCISES K.

Use of squared paper.

(The most convenient paper is that ruled so as to show tenths of an inch. Lines an inch apart should be in a different colour from the others.)

1. Draw various rectangles and count up the little squares. Count up also the divisions in the length and height, and deduce that the area of a rectangle = length × height.

2. Draw various triangles, and count up the number of little squares they contain (counting as 1 square a piece greater than half a square, and omitting portions less than half a square).

Then count up the number of divisions in the base and height, and deduce that

the area of a triangle = ½ base × height.

3. Draw two parallelograms on the same base and of the same height. By counting the little squares show that their areas are equal.

4. Do the same with parallelograms on equal bases and having equal heights.

5. Do the same with triangles.

6. Draw a parallelogram and a triangle on the same base and of the same height. By counting squares, show that the area of the parallelogram is twice the area of the triangle.

7. Draw a right-angled triangle (making two sides along the printed lines), and describe squares on its sides. Find their areas by counting up the little squares, and thus see that the square on the longest side (the hypotenuse) is equal to the sum of the squares on the other sides.

8. Make a wet ink-mark on the rim of a penny, and roll the penny along a printed line. In this way measure the circumference of the circle. Measure also the diameter of the penny.

Divide the circumference by the diameter.

9. Try other coins, and thus see that

Circumference ÷ diameter is always the same.

10. Draw a circle with a diameter along a printed line. Mark a chord along a printed line at right angles to the diameter. See that it is bisected by the diameter. Do this with several circles and chords.

11. Take any two points where printed lines meet, and join them. With the help of your compasses, but without using your ruler, find the length of the line joining the points.

12. A man travels 2 miles West, and then 3 miles North: how far is he then from his starting point?

13. A man travels 1 mile East, then 2 miles North, then 3 miles West, then 5 miles South, and then 1 mile East. How far is he then from his starting point?

14. A path goes straight for 3 yards, then 8 yards to the left at right angles, and then straight to the point midway between the start and the first turn. Find the length of the path.

15. If a ladder is 10 feet long and has its foot 6 ft. from a vertical wall, how high up the wall does it reach?

16. Cut out a figure consisting of two unequal squares with an angle of one adjacent to an angle of the other. By two cuts divide the paper into three pieces which will fit together and form a square.

17. Describe a circle with a radius of 2 inches, and take a point C 1 inch from its centre. Through C draw any two chords DCE, FCH. Measure CD, CE, CF, CH in inches. Multiply the lengths of CD and CE together, and also the lengths of CF and CH. What do you notice about the products? Do this with other circles.

18. On a line AB 4 inches long as diameter describe a circle, and in AB produced take a point C, 1 inch from B. From C draw two straight lines CDE, CFH, cutting the circle at D, E, F, H. Measure CD, CE, CF, CH. Multiply the lengths of CD and CE together, also the lengths of CF and CH. What do you notice about the products? Do this with other circles.

19. A seesaw 10 feet long rests on a pivot 4 feet high. It starts in a horizontal position. Through what angle will one half turn before reaching the ground? How high can the other end go?

20. Draw a square whose diagonal is 2 inches, and describe equilateral triangles on its sides externally. Prove by measurement that a square is formed by joining the vertices of the triangles.

21. A rope runs from the summit of a flagstaff 24 ft. high to a point on the ground 10 ft. from the foot of the flagstaff; find its length.

22. A wall is 24 ft. long and 7 ft. high. Draw it, using $\frac{1}{10}$th of an inch to represent a foot. What is the distance from the top of one end to the bottom of the other?

23. From a sheet of squared paper cut out a rectangle ABCD 3 inches by 4 inches. Fold it so that the point A falls on the point C. Measure the angle between AC and the crease, and also measure the length of the crease.

EXERCISES M.

Areas.

1. Draw a rectangle whose area is 6 square inches.

2. Draw another rectangle of the same area but of a different shape.

3. Draw a rectangle, and on the same base draw a parallelogram of the same altitude. By cutting and superimposing, prove that the areas of the two figures are equal. What do you conclude as to the area of a parallelogram?

4. Draw a parallelogram ABCD, and join AC. By cutting and superimposing, see that the line AC bisects the parallelogram. Using the previous exercise, what do you discover about the area of a triangle?

5. Draw a rectangle whose area is 4·8 sq. inches.

6. Draw another rectangle of the same area but of a different shape.

7. Take any triangle, and fold the base upon itself, so as to get the perpendicular from the vertex to the base. Open the triangle, and fold all three corners to the foot of this perpendicular. Hence deduce that the area of a triangle is equal to one-half the product of base and perpendicular. (Why is the folded figure a rectangle?)

8. By means of the same experiment prove that the three angles of a triangle are equal to two right angles.

9. Also by the same experiment prove that the line joining the middle points of two sides of a triangle is parallel to the third side.

EXERCISES N.

Symmetrical Figures. Lines of Symmetry.

We call a figure symmetrical if, by one folding, we can get the left-hand side to coincide with the right.

1. Make an isosceles triangle, and prove that it is symmetrical. Its line of symmetry is a bisector,—of what?

2. We call two points **images** (each of the other) with respect to a line when a folding about that line brings them together. They are evidently equidistant from the crease. (What is meant by the distance of a point from a line?)

3. The **join** of a point and its image is perpendicular to the line of folding. Verify this by measurement.

In folding a symmetrical figure, the crease is called a **line of symmetry, or an axis.**

4. Make a symmetrical triangle, and mark its axis.

5. How many axes has an equilateral triangle ?

6. Is a rectangle symmetrical about a diagonal ? Is it symmetrical about any line ?

7. In a previous exercise you have seen that the diagonal of a parallelogram divides it into two triangles which are equal in all respects. Is a diagonal a line of symmetry ?

8. Cut out a parallelogram and see if you can discover a line of symmetry. Investigate the case of a rhombus. See Def., p. 61.

9. Describe a circle, and fold it about any diameter. What do you discover as to lines of symmetry ?

10. Prove, by folding, that if a diameter of a circle is at right angles to a chord, it bisects that chord.

11. Also see that the converse is true, viz., if a diameter of a circle bisects a chord, it is at right angles to it.

12. Make a regular hexagon. Find out how many axes it has.

EXERCISES P.

Similar Figures.

1. Draw a triangle ABC, and a line DE parallel to BC, cutting AB at D, and AC at E. By measurement, or otherwise, ascertain that the two triangles are equiangular. (What does equiangular mean ?)

2. Cut out a triangle whose sides are 3, 5, and 7 inches. Also one whose sides are 3, 5, and 7 centimetres. By superposition see that the triangles are equiangular, and therefore of the same shape. *We call such triangles* **similar.**

3. Cut out triangles whose sides are 2, 3, 4 inches and 4, 6, and 8 inches. See by superimposing, that they are similar.

4. A triangle has sides of lengths 3, 4, 6 feet. A similar triangle has its longest side 12 feet long. Find the lengths of its other sides.

5. All equilateral triangles are similar. Prove this experimentally.

6. Take any triangle ABC, and draw DE parallel to BC, cutting AB at D, and AC at E. Prove by measurement that $\frac{AD}{AE}=\frac{AB}{AC}$.

7. In the above, prove that $\frac{AD}{DB}=\frac{AE}{EC}$.

8. Draw a straight line ABC, so that AB = 1 inch, and BC = 2 inches. Draw any other straight line AD. Join CD, and draw BE parallel to CD, meeting AD at E. Find the value of the fraction $\frac{AE}{AD}$.

9. Devise a method for dividing a straight line into 5 equal parts.

10. Divide a given straight line in the ratio of 3 to 4.

11. Divide a given straight line in the ratio of 4 to 5.

12. A rectangle ABCD whose sides are 3 and 4 feet long is folded so that the point A falls on the point C. Find the length of the crease.

13. DE is drawn parallel to the side BC of a triangle ABC, cutting AB at D in the ratio of 3 to 4, and AC at E. Find by measurement the ratio of AE to EC.

14. Do the same for another triangle when D divides AB in the ratio of (1) 3 to 5, (2) 5 to 7, (3) 3 to 8. What do you deduce?

15. What relation do you establish about such triangles as the above?

16. On squared paper make two *similar* parallelograms.

17. On squared paper take two similar triangles. (Take the angular points where printed lines cross one another.) Find the ratio of their areas.

Do this with several pairs of triangles, and hence prove that the areas of similar triangles are in the ratio of the squares of their corresponding sides. (Corresponding sides are sides opposite equal angles.)

BOOK I.

DEFINITIONS.

[The definitions should not be learnt *en bloc*, but only as required in the propositions.]

1. The **volume** of a body is the amount of space it occupies.

A volume has **length**, **breadth**, and **height**, and is therefore said to be of three dimensions.

2. The **surface** of a body is the boundary which separates it from the rest of space.

A surface has length and breadth, but no height or thickness. It is therefore said to be of two dimensions.

3. The intersections of surfaces are called **lines**.

A line has length, but no breadth or thickness. It is therefore said to be of one dimension.

4. The extremities of any portion of a line, or the intersections of lines, are called **points**.

A point has neither length, nor breadth, nor thickness. It is therefore said to have no dimensions.

We can **represent** a line by a fine pencil mark, but we cannot actually draw a line, for we cannot make a pencil mark which has no breadth or thickness.

In the same way we can **represent** a point by a dot, or by the prick of a needle on a piece of paper, but we cannot actually draw a point, for we cannot make a mark which has absolutely no dimensions.

5. A **straight** line is a line which lies evenly between its extreme points.

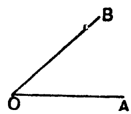
We see, therefore, that the shortest distance between two points is a straight line.

6. A **curve**, or curved line, is a line which is neither straight nor made up of straight lines.

7. A **plane**, or a plane surface, is a surface in which, any two points being taken, the straight line between them lies wholly in that surface.

8. Any collection of points, lines, or surfaces, is called a **figure**.

9. Let a straight line rotate in a plane about the point O; and let it, starting from the position OA, move into the position OB, as shown in the figure. The amount of turning which the line has done in moving from the position OA to the position OB is called a **plane angle**, or shortly, an **angle**.



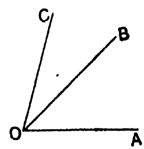
The angle is denoted by the letters AOB.

The point O is called the **vertex** of the angle.

The straight lines OA, OB are called the **arms** of the angle; and they are said to contain the angle AOB.

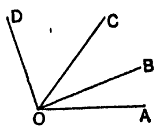
The beginner must carefully notice that the size of an angle in no way depends upon the lengths of its arms.

10. When three straight lines OA, OB, OC are drawn from a point, as in the figure, the angles AOB, BOC are said to be adjacent angles.

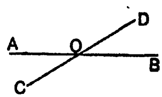


In the above figure, when the straight line OA rotates from the position OA into the position OC, it rotates further than when it rotates from the position OA into the position OB; and hence the angle AOC is said to be greater than the angle AOB.

Also the angle BOC is said to be the difference of the angles AOC, AOB.



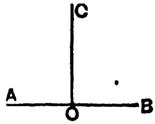
If four straight lines OA, OB, OC, OD meet at a point O (see figure) we see that (1) the angle AOD is equal to the sum of the angles AOB, BOC, COD; (2) the angles AOC, COD are equal to the three angles AOB, BOC, COD, and so on.



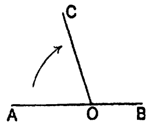
If two straight lines AB, CD intersect at O, the angles AOC, BOD are said to be vertically opposite angles.

11. When one straight line standing on another straight line makes the adjacent angles equal to one another, each of the angles is said to be a right angle, and the straight line standing on the other is called a perpendicular to it.

As OC rotates from the position OA to the position OB, the angle AOC continually **increases**, and the angle BOC continually **decreases**: thus we see that in only one position of OC are the adjacent angles AOC, BOC equal. Therefore through a point in a straight line, one, and only one, straight line can be drawn perpendicular to that straight line.



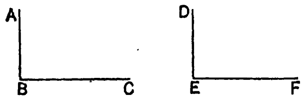
If AOB be a straight line, and OC rotate from OA to OB, in the direction shown by the arrow-head in the figure, it is evident that there must be a position of OC in which the angle AOC is equal to the angle BOC. When OC is in that position each of the angles AOC, BOC is a right angle.



It is generally assumed that **all right angles are equal**, but it is capable of proof as follows.

Let ABC, DEF be right angles.

Apply the angle DEF to the angle ABC, so that the point E falls on the point B, and the straight line EF falls along the



straight line BC. Also let the straight line ED fall on the same side of BC as BA.

Then if ED does not fall along BA, we shall have two straight lines at right angles to BC at the same point, which is impossible. Therefore the right angles ABC, DEF coincide and are equal.

12. An **obtuse** angle is greater than a right angle.

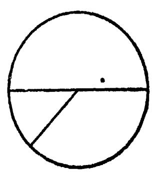
13. An **acute** angle is less than a right angle.

14. A **circle** is a plane figure contained by one line called the circumference, and is such that all straight lines drawn from a certain point within the figure to the circumference are equal to one another.

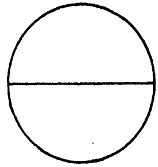
This point is called the **centre** of the circle.

A point is said to be **on the circle** when it lies on the circumference.

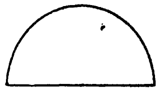
A point is said to be **within the circle** when it lies within the circumference.



15. A **radius** of a circle is a straight line drawn from the centre to the circumference.



16. A **diameter** of a circle is a straight line drawn through the centre of the circle, and terminated both ways by the circumference.

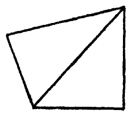


17. A **semicircle** is the figure bounded by a diameter of a circle and the part of the circumference which it cuts off.

18. **Rectilineal** figures are such as are bounded by straight lines.

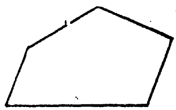
19. A **triangle**, or trilateral figure, is contained by three straight lines.

Any one of the angular points of a triangle may be called a vertex, and the opposite side is then called the base. If two sides are mentioned, the third side is called the base.



20. A **quadrilateral** is a plane figure bounded by **four** straight lines.

A straight line joining opposite angular points of a quadrilateral is called a **diagonal.**



21. A **polygon** is a plane figure contained by more than four straight lines.

A **convex** polygon is one in which each interior angle is less than two right angles.

22. An **equilateral** triangle is one whose three sides are equal.

23. An **isosceles** triangle is one which has two equal sides.

24. A **scalene** triangle is one which has three unequal sides.

25. A **right-angled** triangle is one which has a right angle.

The side opposite the right angle in a right-angled triangle is called the **hypotenuse.**

26. An **obtuse-angled** triangle is one which has an obtuse angle.

27. An **acute-angled** triangle is one which has three acute angles.

28. A magnitude is said to be **bisected** when it is divided into two equal parts.

29. The **bisector of an angle** is the straight line which divides it into two equal angles.

PROBLEMS (OR CONSTRUCTIONS) AND THEOREMS.

Propositions are either **Problems** or **Theorems.**

In a **problem** something is required to be done or made; *e.g.* it may be required to draw a straight line in some particular manner.

In a **theorem** some geometrical truth is established or proved.

The general statement of what is to be proved or done is called the **Enunciation** of the proposition.

The **particular** enunciation is the statement of what is to be done or proved, **with reference to a figure.**

In the enunciation of a theorem the **Hypothesis** is that which is assumed to be true, and the **Conclusion** is that which has to be proved.

Interchange the hypothesis and conclusion, and we have what is called the **Converse** of the proposition; *e.g.* the theorem, 'If two angles of a triangle are equal, the sides opposite them are equal' is the **converse** of this: 'If two sides of a triangle are equal, the angles opposite them are equal.'

In a problem the given conditions are called the **data,** and the things required the **quaesita.**

The letters Q.E.D., which will be found at the end of each theorem, stand for 'Quod erat demonstrandum.'

The letters Q.E.F., which will be found at the end of each problem, stand for 'Quod erat faciendum.'

SYMBOLS AND ABBREVIATIONS.

The following may be used in writing out the propositions :

$\therefore$	*for*	therefore.	diag^l	*for*	diagonal.
$\because$	,,	because.	prop.	,,	proposition.
$=$	,,	is equal to, *or* are equal to, *or* equal to.	def.	,,	definition.
			hyp.	,,	hypothesis.
			str.	,,	straight.
$\angle$	,,	angle.	cons.	,,	construction.
rt. $\angle$	,,	right angle.	pt.	,,	point.
$\triangle$	,,	triangle.	isos.	,,	isosceles.
$\parallel$ or par^l	,,	parallel.	int.	,,	interior.
par^{m}	,,	parallelogram.	ext.	,,	exterior.
sq.	,,	square.	alt.	,,	alternate.
perp^r	,,	perpendicular.	mid.	,,	middle.
rectil.	,,	rectilineal.	equilat.	,,	equilateral.
rect.	,,	rectangle.	quad^l	,,	quadrilateral.
opp.	,,	opposite.	reqd.	,,	required.
adj.	,,	adjacent.	gr.	,,	greater.

AXIOMS (SELF-EVIDENT TRUTHS).

1. *Things which are equal to the same thing are equal to one another.*
2. *If equals be added to equals the wholes are equal.*
3. *If equals be taken from equals the remainders are equal.*
4. *If equals be added to unequals the wholes are unequal.*
5. *If equals be taken from unequals the remainders are unequal.*
6. *Things which are double of the same thing are equal to one another.*
7. *Things which are halves of the same thing are equal to one another.*
8. *Magnitudes which can be made to coincide with one another are equal.*

[The method of placing one magnitude upon another is called the method of **superposition**, and the one magnitude is said to be **applied** to the other.]

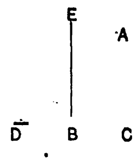
9. *The whole is greater than its part.*
10. *Two straight lines cannot enclose a space.*
11. *All right angles are equal.*
12. **Playfair's Axiom.** *Two intersecting straight lines cannot both be parallel to a third straight line.*

BOOK I.

PROPOSITIONS.

PROPOSITION 1. THEOREM.

If one straight line stands on another straight line, the sum of the adjacent angles so formed is equal to two right angles.



Let the str. line AB meet CD at B.

It is required to prove that the $\angle$*s* ABC, ABD *are together equal to two rt.* $\angle$*s.*

If the $\angle$ABC = the $\angle$ABD,

each is a rt. $\angle$. *Def.*

If the $\angle$ABC be not equal to the $\angle$ABD, suppose BE drawn at rt. $\angle$s to CD at the pt. B.

Then $\angle$CBA + $\angle$ABD = the straight angle CBD

= $\angle$CBE + $\angle$EBD

= 2 rt. angles.

DEFINITION.—When two angles are together equal to two right angles, each is said to be the **supplement** of the other, and the two angles are said to be **supplementary.**

Thus, in the figure of the preceding proposition,

the $\angle$ABC is the supplement of the $\angle$ABD;

and the $\angle$s ABC, ABD are supplementary.

Since all right angles are equal, it follows that if two angles are equal, their supplements are also equal.

DEFINITION.—When two angles are together equal to a right angle, each is said to be the **complement** of the other; and the two angles are said to be **complementary.**

For instance, in the preceding proposition, the angles EBA, ABC are complementary.

EXERCISES.

1. If the angles at the base of a triangle are equal, and the sides are produced, the angles on the other side of the base are equal. II. 1.

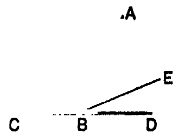
2. If the angles at the base of a triangle are equal, and the base is produced both ways, the exterior angles thus formed are equal. II. 2.

3. If ABC be any angle, and CB be produced to D, the bisectors of the angles ABC, ABD are at right angles. Or: **The internal and external bisectors of an angle are at right angles.** II. 3.

4. A line AB is drawn on a sheet of paper, which is then folded so as to bring A and B together. Show that the crease in the paper is at right angles to AB. II. 4.

PROPOSITION 2. THEOREM.

If the sum of two adjacent angles is equal to two right angles, the exterior arms of the angles are in the same straight line.



At the pt. B in the str. line AB, let the two str. lines CB, DB, on opp. sides of AB, make the adjacent ∠s ABC, ABD together equal to two rt. ∠s.

It is reqd. to prove that BC *and* BD *are in the same str. line.*

For if BD and BC be not in the same str. line,

let BE be in the same str. line with BC.

Then since CBE is a str. line,

the ∠ABE is the supplement of the ∠ABC. (I. 1.)

But the ∠ABD is the supplement of the ∠ABC; *Hyp.*

∴ the ∠ABE = the ∠ABD; ∴ BE falls upon BD.

But CBE is a str. line;

∴ CB and BD are in a str. line. Q.E.D.

State the converse of this proposition.

EXERCISES.

1. If four straight lines meet in a point, and two of the adjacent angles are together equal to the other two; then two of the lines will be in the same straight line. II. 5.

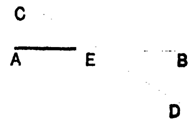
2. AOB is a straight line, and the angles AOP, BOR on opposite sides of it are equal. Prove that POR is also a straight line. II. 6.

3. Four straight lines meet in a point in such a way that opposite angles are respectively equal to one another. Prove that these lines are two, and two, in the same straight line. II. 7.

Proposition 3. Theorem.

If two straight lines cut one another, the vertically opposite angles are equal.

Let the str. lines AB, CD cut one another at the pt. E.



It is reqd. to prove that

(1) the ∠AEC = the ∠BED ; *and*

(2) the ∠CEB = the ∠AED.

CE meets AB at E ;

∴ the ∠AEC is the supplement of the ∠CEB.

Also BE meets CD at E ;

∴ the ∠BED is the supplement of the ∠CEB ;

∴ the ∠AEC = the ∠BED.

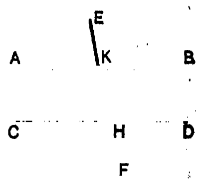
In the same way it may be proved that the ∠CEB = the ∠AED.

Q.E.D.

Cor. 1. *If two straight lines cut one another, the four angles thus formed are together equal to four right angles.*

Cor. 2. *If any number of straight lines meet at a point, the sum of all the angles thus formed is equal to four right angles.*

Definition.—**Parallel** straight lines are straight lines in the same plane which never meet, however far they are produced either way.

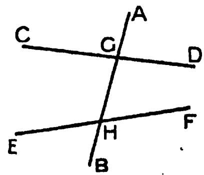


If AB, CD are two parallel straight lines, and the straight line EF cuts them at K and H, the angles AKH, KHD are called **alternate** angles. The angles BKH, KHC are also **alternate**.

The angles AKE, BKE, CHF, DHF are called **exterior** angles, and the angles CHK, KHD, AKH, BKH are the **interior opposite** angles to them respectively.

CORRESPONDING ANGLES.

If a straight line AB cuts the straight lines CD and EF at G and H, the angles AGD, AHF are called **corresponding angles.**



In the same diagram we see that the following pairs of angles also are corresponding angles :

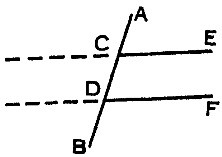
$\angle$AGC and $\angle$AHE;

$\angle$BHE and $\angle$BGC;

$\angle$BHF and $\angle$BGD.

The cutting straight line AB is called a **transversal.**

Take any str. line AB, and at the pts. C and D in it, with the help of your protractor, draw the equal angles ACE, ADF.



If you produce the lines CE, DF in either direction, you will find that they have no tendency to meet.

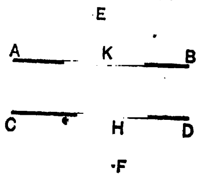
Experiment with different angles, and you will always arrive at the same result. Hence, since the lines CE, DF are in the same plane, they are parallel.

Thus experiment tends to prove that if a transversal AB when cutting two straight lines CE, DF makes the corresponding angles ACE, ADF equal, the lines are parallel.

This is often admitted as axiomatic (self-evident), but a proof is given in the following proposition.

Proposition 4. Theorem.

When a straight line cuts two other straight lines, if two corresponding angles are equal, or if two alternate angles are equal, or if two interior angles on the same side of the transversal are supplementary, the two straight lines are parallel.



Let AB, CD be cut by a transversal EKHF.

(1) *Let the ext.* $\angle$EKB*=the int. opp.* $\angle$KHD.

It is reqd. to prove that AB *and* CD *are parallel.*

If AB, CD are not parallel, they will meet, if produced far enough, say to the right.

Now $\angle$CHF = vert. opp. $\angle$KHD,
and $\angle$AKH = " $\angle$EKB;

$\therefore$ $\angle$CHF = $\angle$AKF,

i.e. ext. $\angle$ equal to int. opp. $\angle$.

$\therefore$ we have the same conditions on the left of EF as we had on the right.

$\therefore$ AB, CD must meet on the left of EF,

i.e. they meet in two points—which is impossible;

$\therefore$ AB and CD are parallel.

(2) *Let it be given that* $\angle$AKH = *alternate* $\angle$KHD.

Then $\angle$EKB = vert. opp. $\angle$AKH = $\angle$KHD,

i.e. the ext. $\angle$ = int. opp. $\angle$;

$\therefore$ by (1) AB and CD are parallel.

(3) *Let it be given that the* $\angle$*s* BKH, KHD *are supplementary.*

Now, because EKH is a str. line, $\angle$EKB is the supplement of $\angle$BKH.

$\therefore$ $\angle$EKB = $\angle$KHD,

i.e. the ext. $\angle$ = int. opp. $\angle$.

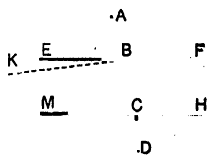
$\therefore$ AB and CD are parallel. Q.E.D.

Playfair's Axiom.—*Two intersecting straight lines cannot both be parallel to a third straight line.*

PROPOSITION 5. THEOREM.

If a straight line fall on two parallel straight lines, it makes (1) *the alternate angles equal,* (2) *the exterior angle equal to the interior opposite angle on the same side of the line, and* (3) *two interior angles on the same side together equal to two right angles.*

Let the str. line ABCD fall on the two ∥ str. lines EF, MH.



It is reqd. to prove that

(1) *the* $\angle$EBC = *the alt.* $\angle$BCH,

(2) *the ext.* $\angle$ABF = *the int. opp.* $\angle$BCH,

(3) *the two int.* $\angle$*s* FBC, BCH *are equal to two rt.* $\angle$s.

(1) If the $\angle$ EBC be not equal to the $\angle$ BCH,

let the $\angle$ KBC be equal to the $\angle$ BCH.

These are alternate $\angle$s;

$\therefore$ KB is $\parallel$ to MH; (I. 4.)

$\therefore$ two intersecting str. lines KB and EB are each $\parallel$ to MH; which is impossible; (Playfair's Axiom.)

$\therefore$ the $\angle$ EBC cannot be unequal to the $\angle$ BCH,

i.e. the $\angle$ EBC = the $\angle$ BCH.

(2) It has been proved that the $\angle$ BCH = the $\angle$ EBC.

Also the $\angle$ ABF = the vertically opp. $\angle$ EBC; (I. 3.)

$\therefore$ the $\angle$ ABF = the $\angle$ BCH.

(3) The $\angle$ CBF = the supplement of the $\angle$ ABF (I. 1.)

= the supplement of the $\angle$ BCH, by the second part of the proposition,

i.e. the $\angle$s CBF and BCH = two rt. $\angle$s. Q.E.D.

NOTE.—References to propositions by *numbers* are only inserted to enable learners to refer back for verification of statements.

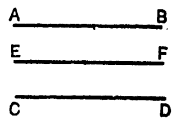
The *numbers* of propositions are not intended to be quoted in examination: but verbal explanation should be given where this can be done concisely: *e.g.* (vert. opp.) to explain why two angles are equal, or (alternate $\angle$s) where two angles are equal because they are between parallels.

PROPOSITION 6. THEOREM.

Straight lines which are parallel to the same straight line are parallel to one another.

Let the two str. lines AB, CD each be parallel to EF.

It is reqd. to prove that AB *is* $\parallel$ *to* CD.



If AB is not $\parallel$ to CD, they will meet; and we then have two intersecting str. lines both $\parallel$ to a third str. line, which, by Playfair's axiom, is impossible.

$\therefore$ AB is $\parallel$ to CD. Q.E.D.

EXERCISES.

1. If CA, CB are respectively parallel to ED and EF, prove that the angle ACB is equal to the angle DEF if both these angles are acute, or both obtuse.

2. In the above, what is the relation between the two angles if one is acute and the other obtuse?

3. ABCD is a quadrilateral such that AB is parallel to CD, and AD to BC. Prove that the angle ABC is equal to the angle ADC.

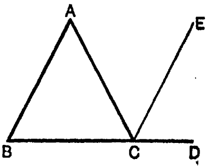
PROPOSITION 7. THEOREM.

If a side of a triangle be produced, (1) *the exterior angle is equal to the two interior opposite angles, and* (2) *the three angles of every triangle are together equal to two right angles.*

In the $\triangle$ABC produce BC to D.

It is reqd. to prove (1) *that the ext.* $\angle$ACD *is equal to the two int. opp.* $\angle s$ BAC, ABC,

and (2) *that the three* $\angle s$ ABC, BAC, ACB *are together equal to two rt.* $\angle s$.



Suppose CE drawn || to BA through the pt. C.

PART I. AB is || to CE ;

∴ the $\angle$ACE = the alt. $\angle$BAC. (I. 5.)

Also the ext. $\angle$ECD = the int. opp. $\angle$ABC ; (I. 5.)

∴ the whole $\angle$ACD = the $\angle$s BAC, ABC.

PART II. The $\angle$ACD = the $\angle$s BAC, ABC. *Proved above.*

Add the $\angle$ACB to each ;

∴ the $\angle$s ACD, ACB = the $\angle$s BAC, ABC, ACB.

But the $\angle$s ACD, ACB = two rt. $\angle$s ; (I. 1.)

∴ the $\angle$s BAC, ABC, ACB = two rt. $\angle$s. Q.E.D.

COR. 1. *If one side of a triangle is produced, the exterior angle is greater than either of the interior non-adjacent angles.*

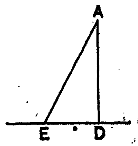
COR. 2. *Any two angles of a triangle are together* LESS THAN *two right angles.*

COR. 3. *Only one perpendicular can be drawn to a straight line from a given point without it.*

For, if possible, let AD and AE each be perpendicular to the straight line DE.

Then in the triangle ADE, the two angles ADE, AED are together equal to two right angles : which is impossible.

Therefore, only one perpendicular, etc. Q.E.D.

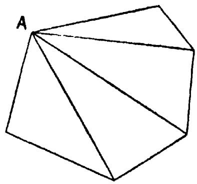


COR. 4. *In a right-angled triangle the right angle is the greatest angle, and the sum of the remaining angles is equal to a right angle.*

PROPOSITION 8. THEOREM.

All the interior angles of any rectilineal figure, together with four right angles, are equal to twice as many right angles as the figure has sides.

Join one vertex A of the rectilineal figure to each of the others.



Each side, except the two ending at A, forms a triangle with A.

$\therefore$ $n-2$ triangles are formed, if the figure has n sides.

Each of these triangles contains angles whose sum is two right angles.

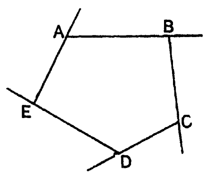
$\therefore$ the sum of all the int. angles of the figure $=2(n-2)$ right angles $=2n$ rt. $\angle$s -4 rt. $\angle$s.

$\therefore$ all the interior angles of the figure, together with four right angles, are equal to $2n$ right angles. Q.E.D.

COR. *All the exterior angles of any convex polygon are together equal to four right angles.*

Let ABCDE be any convex polygon, and produce its sides in the directions AB, BC, CD, DE, EA.

The int. and ext. $\angle$s at each angular pt. are equal to two rt. $\angle$s; (I. 1.)



$\therefore$ all the int. $\angle$s with all the ext. $\angle$s
$=$ twice as many rt. $\angle$s as the figure has sides,
$=$ all the int. $\angle$s with four rt. $\angle$s;
$\therefore$ all the ext. $\angle$s $=$ four rt. $\angle$s.

Q.E.D.

To find the magnitude of the angle of a regular pentagon.

[A polygon of five sides is called a **pentagon**, and a *regular figure* is one which is equilateral and equiangular.]

All the five angles $+4$ rt. $\angle$s $=10$ rt. $\angle$s;

$\therefore$ all the five angles $=6$ rt. $\angle$s;

$\therefore$ each angle $=\frac{6}{5}$ of a rt. $\angle$.

Similarly, the angle of a regular hexagon (a regular polygon having six sides) is $\frac{4}{3}$ of a rt. $\angle$. Thus, three regular hexagons can be placed so as to fit round a point without gaps and without overlapping; for the three angles at the point are together equal to 4 rt. $\angle$s.

EXERCISES.

1. The four interior angles of a quadrilateral are equal to four right angles. XIII. 12.

2. If two angles of a triangle be complementary, the triangle is right-angled. XIII. 13.

3. If one angle of a triangle be equal to the sum of the other two, the triangle is right-angled. XIII. 14.

4. If one angle of a triangle be greater than the sum of the other two, the triangle is obtuse-angled. XIII. 15.

5. Prove that the acute angle between two straight lines is equal to the acute angle between any two straight lines at right angles to them. XIII. 18.

6. If the base BC of the triangle ABC be produced to D, the angle between the bisectors of the angles ABC, ACD is equal to half the angle BAC. XIII. 25.

7. If all the angles of a rectilineal figure are together equal to eight right angles, how many sides has it? XIII. 1.

8. What is the number of sides of a regular polygon if each of its internal angles is 150°? XIII. 2.

9. What fraction of a right angle is

(1) The angle of a regular octagon (an eight-sided polygon),

(2) The angle of a regular decagon (a ten-sided polygon),

(3) The angle of a regular heptagon (a seven-sided polygon)? XIII. 3.

10. The exterior angles of a rectilineal figure (formed by producing the sides successively) are together equal to the interior angles: find the number of sides. XIII. 4.

11. What fractions of a right angle will the angles of a pentagon be if they are in the ratios of the numbers 1, 3, 6, 9, 11? XIII. 5.

12. The five sides of a rectilineal figure ABCDE are equal to one another, and also the five angles; show that the bisector of the angle ABE is at right angles to BC. XIII. 7.

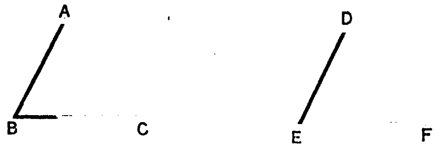
13. In a quadrilateral whose opposite sides are parallel, prove that the straight lines which bisect the angles adjacent to one of its sides are at right angles to one another. XIII. 8.

14. How many of the exterior angles of a triangle must be obtuse? XIII. 10.

15. If two triangles have two angles of the one together equal to two angles of the other, the third angles will be equal. XIII. 11.

PROPOSITION 9. THEOREM.

If two triangles have two sides of one equal respectively to two sides of the other and the included angles equal, the triangles are equal in all respects.



Let ABC, DEF be two $\triangle$s such that

(1) $AB = DE$;

(2) $AC = DF$;

and (3) the included $\angle BAC$ = included $\angle EDF$.

It is reqd. to prove that the $\triangle$s are equal in all respects.

Apply the $\triangle ABC$ to the $\triangle DEF$, placing A on D and AB along DE.

Then AC falls along DF since $\angle BAC = \angle EDF$.

Also AB falling on DE, B must fall on E (since $AB = DE$);

and AC falling on DF, C must fall on F (since $AC = DF$);

$\therefore$ $\triangle ABC$ coincides with $\triangle DEF$;

$\therefore$ $\triangle ABC$ is equal to $\triangle DEF$ in all respects;

i.e.
(1) $BC = EF$,

(2) $\angle ABC = \angle DEF$,

(3) $\angle ACB = \angle DFE$,

(4) the $\triangle$s are equal in area. Q.E.D.

Note that each triangle has six elements or parts, three angles and three sides. It also has an area.

EXERCISES.

1. ABCDE is a regular pentagon; prove that AD is equal to AC. [A polygon of five sides is called a pentagon.] III. 1.

2. In a regular hexagon ABCDEF, prove that AC is equal to DF. [A polygon of six sides is called a hexagon.] III. 2.

3. The straight line DC meets the straight line AB at its middle point C, and is at right angles to AB. Prove that DA is equal to DB. III. 3.

4. In a regular hexagon ABCDEF, prove that AEC is an equilateral triangle. III. 4.

5. Two straight lines AB, CD bisect one another at E. Prove that the triangles AEC, BED are equal in all respects. III. 5.

6. Two quadrilaterals ABCD, XYZW have AB = XY, BC = YZ, CD = ZW, ∠B = ∠Y, ∠C = ∠Z. Show that they are equal in all respects. III. 7.

7. A and B are two points on the same side of the straight line joining two other points C, D. Show that if AC is equal to BD, and the angle ACD to the angle BDC, A and B are equidistant from the middle point of CD. III. 8.

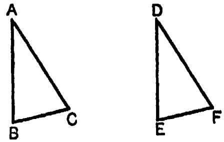
8. With the vertex A of an isosceles triangle ABC as centre, a circle is described which cuts the equal sides AB, AC at D and E respectively. Prove that the triangles ACD, ABE are equal in all respects. III. 9.

9. On a straight line AB, 2 inches long, make an equilateral triangle ABC. If CD bisects the angle ACB, and meets AB at D, what can you discover about the lengths of AD and BD? III. 11.

10. AOB is a straight line, 2 inches long, and O its middle point. Make angles AOC, BOD each equal to 60°, and cut off OC and OD each equal to 3 inches. Prove that AC = BD. III. 12.

Proposition 10. Theorem.

If two triangles have two angles of the one equal to two angles of the other, each to each, and one side equal to one side, namely, either the sides adjacent to the equal angles in each triangle, or sides opposite to equal angles; then the triangles are equal in all respects.



Let the △s ABC, DEF be such that

(*a*) (1) the ∠ABC = the ∠DEF,

(2) the ∠ACB = the ∠DFE,

(3) BC = EF (sides adjacent to equal ∠s in each △).

or (*b*) (1) the ∠ABC = the ∠DEF,

(2) the ∠BAC = the ∠EDF,

(3) BC = EF (sides opp. to equal ∠s in each △).

It is reqd. to prove that the △s ABC, DEF are equal in all respects.

The three ∠s of a △ are together equal to two rt. ∠s;

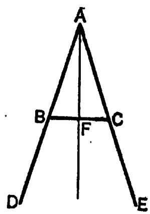
∴ in both cases, the third angles are equal.

Apply the $\triangle ABC$ to the $\triangle DEF$, so that B falls on E, and BC on EF.

Then C coincides with F, for $BC = EF$;
and BA falls along ED, for the $\angle ABC =$ the $\angle DEF$.
Also CA falls along FD, for the $\angle BCA =$ the $\angle EFD$;
$\therefore$ the point A coincides with the point D,
i.e. the $\triangle$s coincide with one another,
and are therefore equal in all respects. Q.E.D.

PROPOSITION 11. THEOREM.

The angles at the base of an isosceles triangle are equal, and if the equal sides be produced, the angles on the other side of the base are equal.



Let ABC be an isosceles $\triangle$, having AB equal to AC.

It is reqd. to prove that

(1) $\angle ABC = \angle ACB$,

and (2) if AB, AC be produced to D, E respectively,

$\angle DBC = \angle ECB$.

Let AF be the bisector of the $\angle BAC$, and let it meet BC at F.

Then in $\triangle$s BAF, CAF,

(1) $AB = AC$, *Given.*
(2) AF is common,
(3) included $\angle BAF =$ included $\angle CAF$;
$\therefore$ $\angle ABF = \angle ACF$. (I. 9.)

Also $\angle DBC =$ the supplement of $\angle ABC$
= „ „ „ $\angle ACB$
$= \angle ECB$. Q.E.D.

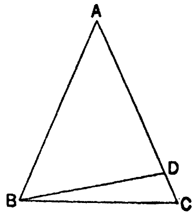
Alternative Proof.

Fold the $\triangle$ about the bisector AF.
Then AE falls along AD, for $\angle CAF = \angle BAF$. *Given.*
Also the pt. C falls on the pt. B, for $AC = AB$;
$\therefore$ FC coincides with FB;
$\therefore$ $\angle ACF$ coincides with $\angle ABF$, and is equal to it.
Also the $\angle$s ECB, DBC coincide and are equal. Q.E.D.

PROPOSITION 12. THEOREM.

(Converse of Proposition 11.)

If two angles of a triangle are equal, the sides opposite to them are equal.



Let ABC be a $\triangle$ having the $\angle$ABC equal to the $\angle$ACB.

It is reqd. to prove that AB = AC.

If AB be not equal to AC, one of them must be the greater.

Let AC be the greater, and suppose that AD = AB. Join BD.

Then the $\angle$ABD = the $\angle$ADB, for AD = AB. (I. 11.)

But the $\angle$ABC is gr. than the $\angle$ABD;

$\therefore$ the $\angle$ABC is gr. than the $\angle$ADB.

But the ext. $\angle$ADB is gr. than the int. opp. $\angle$DCB;

(I. 7. *Cor.* 1.)

$\therefore$ the $\angle$ABC must be gr. than the $\angle$BCD.

But, by hypothesis, the $\angle$ABC = the $\angle$BCD;

$\therefore$ AB and AC cannot be unequal;

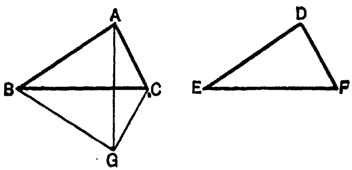
$\therefore$ AB = AC. Q.E.D.

Exercise. Prove the above proposition by folding the triangle about E the middle point of BC, so that EC falls along EB.

Also prove the proposition by folding the triangle about the bisector of the vertical angle.

PROPOSITION 13. THEOREM.

If two triangles have the three sides of the one equal to the three sides of the other respectively, they are equal in all respects.



Let ABC, DEF be the two $\triangle$s, having

(1) AB = DE, (2) BC = EF, (3) AC = DF.

It is reqd. to prove that the $\triangle$s are equal in all respects.

Of the sides of the $\triangle ABC$, let BC be that which is not less than either of the other two.

Apply the $\triangle DEF$ to the $\triangle ABC$, so that the equal bases EF and BC coincide, E with B and F with C, the equal sides being then conterminous, and the vertices falling on opposite sides of the base BC.

Let GBC represent the $\triangle$ so applied, so that G is the position of the point D. Join AG.

$BA = BG$, by hypothesis; $\therefore$ $\angle BAG = \angle BGA$. (I. 11.)

Also $CA = CG$, by hypothesis; $\therefore$ $\angle CAG = \angle CGA$; (I. 11.)

$\therefore$ the whole $\angle BAC =$ the whole $\angle BGC$,

i.e. $\angle BAC = \angle EDF$.

Then in the $\triangle$s BAC, EDF,

(1) $AB = DE$,

(2) $AC = DF$,

(3) included $\angle BAC =$ included $\angle EDF$;

$\therefore$ $\triangle ABC = \triangle DEF$ in all respects. (I. 9.)

Q.E.D.

NOTE.—*Triangles equal in all respects, or identical equality of triangles.*

From Propositions 9, 10 and 13 we see that two triangles are equal in all respects, *i.e.* congruent, when the following three independent elements, or parts, are severally equal:

(1) Two sides and the included angle. (I. 9.)

(2) Two angles and the adjacent side. (I. 10.)

(3) Two angles and the side opposite one of them. (I. 10.)

(4) The three sides. (I. 13.)

We see that in each case we have at least one *side* equal to one side.

Two triangles which have their angles severally equal are not necessarily equal in all respects. In fact, here we are not given three *independent* equalities, for if two angles of a triangle are known, the third is also known, because the sum of the three is equal to two right angles.

EXERCISES.

1. If two sides of a triangle, when produced, make the exterior angles equal, prove that the triangle is isosceles. V. 1.

2. A triangle that has all its angles equal to one another is equilateral. V. 2.

3. The straight lines bisecting the angles ABC, ACB of a triangle ABC intersect at O. Prove that if OB is equal to OC, the triangle ABC must be isosceles. V. 3.

4. The angles DAB, ABC of a quadrilateral ABCD are equal to one another, and also the angles BCD, CDA. Prove that the side AD is equal to the side BC. V. 4.

5. In AB, AC the sides of a triangle ABC, points D, E are taken such that the angles DCB, EBC are equal. If BE, CD are also equal, show that the triangle ABC is isosceles. V. 5.

6. In a regular pentagon ABCDE, prove that the triangle EBC is isosceles. V. 7.

[A polygon of five sides is called a **pentagon**.]

7. Draw a straight line AB 4 cms. long, and make the angles CAB, CBA each equal to 40°. What do you find about the lengths of CA and CB ? V. 10.

EXERCISES.

1. CAD, CBD are two isosceles triangles upon the same base CD, but upon opposite sides of it. Show that the angles CAD, CBD are bisected by the straight line AB. VI. 1.

2. With centres A and B two circles are drawn intersecting in C and D. If AB and CD meet at E, prove that the triangles AEC, AED are equal in all respects. VI. 2.

3. The opposite sides AB, CD of a quadrilateral ABCD are equal, and the straight lines bisecting AD, BC at right angles meet in O. Show that the triangles OAB, ODC are equal in all respects. VI. 3.

4. ABC is an equilateral triangle; a point G is taken within the triangle such that the angle GBC is equal to the angle GCB, and the line AG is drawn. Prove that AG bisects the angle BAC. VI. 4.

5. On the circumference of a circle three points A, B, C are taken so that the straight lines AB, BC are equal. Prove that the straight line joining B with the centre of the circle bisects the angle ABC, and cuts the straight line AC at right angles. VI. 5.

6. If a triangle ABC be turned over about its side AB, show that the line joining the two positions of C is perpendicular to AB. VI. 6.

7. The sides BA, CA of a triangle are produced towards A to D and E respectively, so that BD is equal to CE. If DE is equal to BC, show that the angle EDB is equal to the angle ECB. VI. 7.

8. ABC, ABD are two triangles on the same side of AB, having their three sides respectively equal. If AD, BC meet in O, prove that the triangles AOC, BOD are equal in all respects. VI. 8.

9. A, B, C, D are four points on a circle whose centre is O, and the angle AOB is equal to the angle COD, OB lying between OA and OC, and OC lying between OB and OD. Prove that a point whose distances from A and D are equal is also equidistant from B and C. VI. 9.

10. ABCD is a quadrilateral with its sides AB, AD equal, and the angles ABC, ADC equal. Show that AC bisects BD at right angles. VI. 10.

11. A straight line AB is bisected at C, and on the same side of the line two triangles CAD, CBE are described, having the two sides CD, DA equal to the two sides CE, EB, each to each. Prove that the lines AE, BD are equal. VI. 11.

12. Two equal straight lines AB and CD are joined towards opposite parts by the equal straight lines AD and CB intersecting at O. Prove that both triangles OAC, OBD are isosceles. VI. 12.

13. A point O is taken within an equilateral four-sided figure ABCD such that its distances from the angular points A and C are equal. Show that OB and OD are in one and the same straight line. VI. 13.

14. AOB, COD are two intersecting straight lines, and each of the figures AOCE, BODF is equilateral. Show that the straight line EF passes through O. VI. 14.

15. From the extremities A, B of the base of a triangle are drawn straight lines AD, BE which intersect in O and cut the opposite sides in D, E. Show that, if OA is equal to OB, and OD is equal to OE, the triangle is isosceles. VI. 15.

16. ABC is a triangle; BD, CE are drawn making equal angles with BC, and meeting the opposite sides in D and E, and each other in F. Prove that if the angle AFE is equal to the angle AFD the triangle is isosceles. VI. 16.

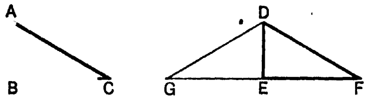
17. Make an isosceles triangle ABC, having AB = AC = 7 cm., and BC = 5 cm. If D is the middle point of BC, what can you prove about the angles ADB, ADC? VI. 17.

18. Make AOB an equilateral triangle having each of its sides 4 cm. long. Produce AO, BO to C and D, making OC = OD = 6 cm. Take E the middle point of AB, and joining EO, produce it to meet CD at F. You will see that DF = CF. Prove that this must be the case if the figure is drawn correctly. VI. 18.

19. Draw a straight line AOB, such that AO = OB = 4 cm. With centre A and radius 7 cm. describe a circle; with centre B and the same radius describe another circle. If D is one of the points of intersection of these circles, prove that the angles DOA, DOB are right angles. What problem have you solved? VI. 19.

PROPOSITION 14. THEOREM.

Two right-angled triangles which have their hypotenuses equal, and one side of the one equal to one side of the other, are equal in all respects.



Let ABC, DEF be two $\triangle$s right-angled at B and E, and such that

AC = DF and AB = DE.

It is reqd. to prove that the $\triangle$s are equal in all respects.

Apply the $\triangle$ABC to the $\triangle$DEF so that AB may coincide with its equal DE, and C may fall on the side of DE opposite to F.

Let G then be the position of the pt. C.

Each of the $\angle$s DEF, DEG is a rt. $\angle$;

$\therefore$ GEF is a str. line. (I. 2.)

In the $\triangle$DGF, DF = DG (or AC);

$\therefore$ $\angle$DGF = $\angle$DFG. (I. 11.)

Hence in the $\triangle$s DEG, DEF,

(1) $\angle$DEG = $\angle$DEF,

(2) $\angle$DGE = $\angle$DFE. *Proved above.*

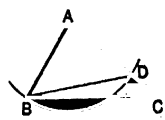
(3) DE is common;

$\therefore$ the $\triangle$s DEG, DEF are equal in all respects, (I. 10.)

i.e. the $\triangle$s ABC, DEF are equal in all respects. Q.E.D.

PROPOSITION 15. THEOREM.

If two sides of a triangle are unequal, the angle opposite the greater side is greater than the angle opposite the less.



Let ABC be a $\triangle$ having AC gr. than AB.

It is reqd. to prove that $\angle$ABC *is gr. than* $\angle$ACB.

With centre A and rad. AB describe a circle cutting AC in D.

The pt. D falls in AC, for AC is gr. than AB.

Join BD.

AD = AB (radii);

$\therefore$ $\angle$ABD = $\angle$ADB. (I. 11.)

But the ext. $\angle$ADB is gr. than the int. opp. $\angle$ACB; (I. 7, *Cor.* 1.)

$\therefore$ also $\angle$ABD is gr. than $\angle$ACB;

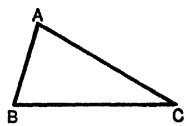
$\therefore$ $\angle$ABC is gr. than $\angle$ACB. Q.E.D.

NOTES.—This proposition is often enunciated thus:

The greater side of every triangle has the greater angle opposite to it. In this form, beginners often fail to see what is *assumed*, and what is *to be proved*. It is very important to remember in this, as in other cases, that the hypothesis is placed *first*.

PROPOSITION 16. THEOREM.

If two angles of a triangle are unequal, the side opposite the greater angle is greater than the side opposite the less.



Let ABC be a $\triangle$, having $\angle$ABC gr. than $\angle$ACB.

It is reqd. to prove that AC *is gr. than* AB.

For if AC is not gr. than AB,

then either (1) AC = AB,

or (2) AC is less than AB.

(1) If AB = AC,

$\angle$ACB = $\angle$ABC, (I. 11.)

which, by hyp., is not the case.

(2) If AC is less than AB,

$\angle$ABC is less than $\angle$ACB, by the previous prop., (I. 15.)

and this, also by hyp., is not the case;

$\therefore$ AC is neither equal to nor less than AB;

$\therefore$ AC is gr. than AB. Q.E.D.

NOTES.—This proposition is often enunciated thus:

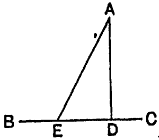
The greater angle of every triangle is subtended by the greater side.

Here, as in Proposition 15, it is important to remember that the hypothesis is placed *first*.

The method of proof used here is commonly known as the **Proof by Exhaustion.**

Proposition 17. Theorem.

Of all straight lines drawn from a given point to a given straight line, the perpendicular is the shortest.



Let AD be perp^r to the str. line BC.

It is reqd. to prove that AD *is shorter than any other str. line* AE *drawn from the pt.* A *to the str. line* BC.

The three angles of the $\triangle$ADE = two rt. $\angle$s;

$\therefore$ since $\angle$ADE = one rt. $\angle$,

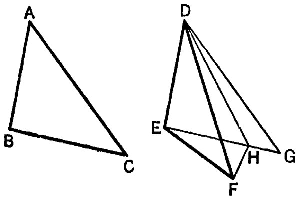
$\angle$AED is less than a rt. $\angle$, *i.e.* $\angle$ADE is gr. than $\angle$AED;

$\therefore$ the side AE is gr. than the side AD,

which proves the proposition.

Proposition 18. Theorem.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of the one greater than the angle contained by the two sides of the other; then the base of that which has the greater angle is greater than the base of the other.



Let ABC, DEF be two $\triangle$s, such that

(1) AB = DE, (2) AC = DF,

(3) $\angle$BAC is gr. than $\angle$EDF.

It is reqd. to prove that the base BC *is gr. than the base* EF.

Apply the $\triangle$ABC to the $\triangle$EDF so that the pt. A falls on D, and the str. line AB on DE.

Then B coincides with E, for AB = DE.

Let C fall at G.

(1) If EG passes through F, then EG is gr. than EF, and the proposition is proved.

(2) But if not, let DH, the bisector of the angle FDG, meet EG at H. Join FH.

Then in $\triangle$s FDH, GDH,

(1) DF = DG, *Given.*

(2) DH is common,

(3) $\angle$FDH = $\angle$GDH ;

$\therefore$ FH = GH. (I. 9.)

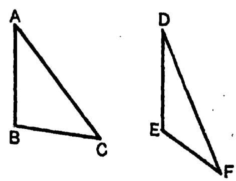
Again in $\triangle$EFH, EH and HF are together gr. than EF ;

$\therefore$ EH and HG are gr. than EF,

i.e. EG (or BC) is gr. than EF. Q.E.D.

Proposition 19. Theorem.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the base of the one greater than the base of the other; then the angle contained by the sides of that which has the greater base is greater than the angle contained by the sides of the other.



Let ABC, DEF be two $\triangle$s, such that

(1) AB = DE,

(2) AC = DF,

(3) BC is gr. than EF.

It is reqd. to prove that $\angle$BAC *is gr. than* $\angle$EDF.

For, if $\angle$BAC be not gr. than $\angle$EDF,

either (1) $\angle$BAC = $\angle$EDF, or (2) $\angle$BAC is less than $\angle$EDF.

(1) If $\angle$BAC = $\angle$EDF,

in $\triangle$s ABC, DEF

(1) AB = DE,

(2) AC = DF,

(3) included $\angle$BAC = included $\angle$EDF ;

$\therefore$ BC = EF, (I. 9.)

which is contrary to the hypothesis.

(2) If $\angle$BAC is less than $\angle$EDF,

BC is less than EF, (I. 18.)

which also is contrary to the hypothesis ;

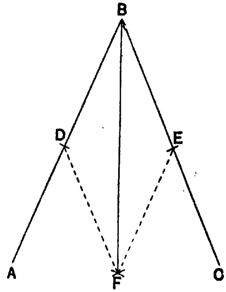
$\therefore$ $\angle$BAC is gr. than $\angle$EDF. Q.E.D.

Proposition 20. Problem.

To bisect a given angle; that is, to divide it into two equal angles.

Let ABC be the given angle.

It is required to bisect it.



With centre B, and any radius, describe a circle cutting BA and BC in D, E respectively.

With centre D, and the same radius, describe a circle; and with centre E and the same radius describe another circle.

Let F be one pt. of intersection of these last two circles, the other being B. Join BF.

BF *will bisect the* ∠ABC. *Join* DF, EF.

In the △s DBF, EBF, (1) BD = BE (radii),

(2) BF is common,

(3) FD = FE (for each = DB); *Cons.*

∴ the ∠DBF = the ∠EBF, (I. 13.)

i.e. BF bisects the ∠ABC. Q.E.F.

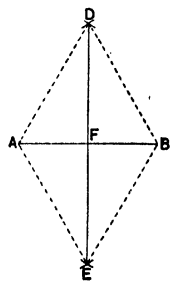
Exercise. Draw a right angle by means of a set-square, and bisect it.

Proposition 21. Problem.

To bisect a given finite straight line.

Let AB be the given finite str. line.

It is reqd. to bisect it.



With centres A and B, and any convenient (but the same) radius, describe circles intersecting at D and E.

(A radius somewhat greater than half AB is convenient.)

Join DE, cutting AB in F.

AB shall be bisected at F. Join AD, AE, BD, BE.

In the △s DAE. DBE, (1) DA = DB, *Cons.*

(2) DE is common,

(3) AE = BE; *Cons.*

∴ the ∠ADE = the ∠BDE. (I. 13.)

Again in the Δs DAF, DBF,

(1) DA = DB,

(2) DF is common,

(3) the included $\angle$ADF = the included $\angle$BDF; *Proved above.*

$\therefore$ AF = CF, (I. 9.)

i.e. AB is bisected at F. Q.E.F.

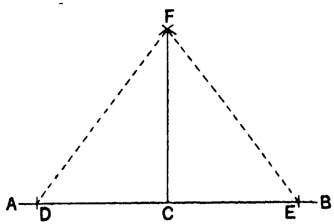
EXERCISES.

1. Draw a str. line 3·7 in. long, and bisect it geometrically.

2 „ 6·9 cm. „ „ „

PROPOSITION 22. PROBLEM.

To draw a straight line at right angles to a given straight line from a given point in it.



Let AB be the given str. line, and C the given pt. in it.

It is reqd. to draw from C *a str. line at rt.* $\angle$*s to* AB.

With centre C, and any radius, describe a circle cutting AB in D and E.

With centre D, and any radius greater than CD or CE, describe a circle.

With centre E, and the same radius, describe a circle.

Take F one of the pts. of intersection of these last two circles.

Join CF.

CF shall be at rt. $\angle$s to AB.

Join DF, EF.

In the △s DFC, EFC,

(1) DC = CE (radii),

(2) CF is common,

(3) DF = EF (radii of equal circles) ; *Cons.*

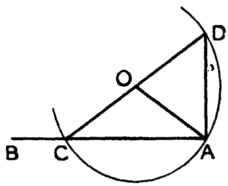
∴ the ∠DCF = the ∠ECF. (I. 13.)

But these are adjacent angles;

∴ CF is at rt. ∠s to AB. Q.E.F.

PROPOSITION 23. PROBLEM.

To draw a straight line, perpendicular to a given straight line, from a point at its extremity.



Let AB be a str. line.

It is reqd. to draw a str. line perp^r *to* BA *at its extremity* A.

Take any point O, not in AB, and with centre O and radius OA describe a circle cutting AB again at C.

Join CO and produce it to meet the circle again at D.

Join AD.

AD will be perp^r to AB.

Join OA.

OA = OD ; ∴ ∠OAD = ∠ODA.

OA = OC ; ∴ ∠OAC = ∠OCA ;

∴ by addition, ∠CAD = ∠ODA + ∠OCA,

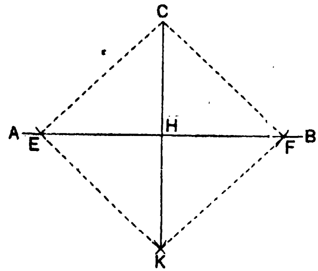
i.e. ∠CAD = one half of the ∠s of the △CAD

= one half of two rt. ∠s

= a rt. ∠. Q.E.F.

PROPOSITION 24. PROBLEM.

To draw a perpendicular to a given straight line of unlimited length from a given point outside it.



Let AB be the given str. line, and C the given pt. outside it.

It is reqd. to draw from C *a perpendicular to* AB.

With centre C describe a circle EDF cutting AB at E and F.

With centres E and F, and the same rad. as before, describe other circles.

These circles pass through C.

Let K be their other pt. of intersection.

Join CK, cutting AB at H.

CK will be perpendicular to AB. Join EC, EK, FC, FK.

In the $\triangle$s ECK, FCK, (1) CE = CF, *Radii.*

(2) CK is common,

(3) EK = FK; *Radii of equal circles.*

$\therefore$ the $\angle$ECK = the $\angle$FCK. (I. 13.)

Again, in the $\triangle$s ECH, FCH,

(1) CE = CF,

(2) CH is common,

(3) the included $\angle$ECH = the included $\angle$FCH; *Proved above.*

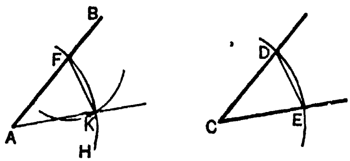
$\therefore$ the $\angle$EHC = the $\angle$FHC, (I. 9.)

and being adj., these $\angle$s are right angles. Q.E.F.

Exercise. Draw a $\triangle$ABC such that AB = 10 cm., AC = 8 cm., BC = 6 cm. From C draw CD perp[r] to AB, and measure it.

PROPOSITION 25. PROBLEM.

At a given point in a straight line to make an angle equal to a given angle.



Let A be the given pt. in the given str. line AB, and DCE the given ∠.

It is required to make at the pt. A *in the line* AB *an* ∠ *equal to the* ∠ DCE.

With centre C, and any radius, describe a circle cutting CD and CE in D and E respectively. Join DE.

With centre A, and the same radius, describe a circle FKH cutting AB in F.

With centre F, and radius equal to DE, describe a circle cutting the circle FKH in K. Join AK;

the ∠FAK will be the ∠ reqd.

Join FK.

In the △s FAK, DCE,

(1) AF = CD. (2) AK = CE, (3) FK = DE; *Cons.*

∴ the ∠FAK = the ∠DCE. (I. 13.)

Q.E.F.

Note that this construction will give two angles, each equal to ∠DCE: one on each side of AB.

EXERCISES.

1. On a given straight line AB 3·5 in. long describe an equilateral triangle. XV. 1.

2. On AB 1·7 in. long as base describe an isosceles triangle, having each of its sides double of AB. XV. 2.

3. On a given base 2·6 in. long describe an isosceles triangle, having each of its sides 1·9 in. long. XV. 3.

4. Describe a circle of given radius, which shall pass through two given points. When is the problem incapable of solution? XV. 4.

5. Find a point equidistant from two given points A and B. XV. 5.

6. With a ruler and compasses, make a right angle. State your construction, and give a proof. XV. 6.

7. In a given straight line find a point which is equidistant from two given points, neither of which lies in the line; and explain in what case the problem is impossible. XV. 7.

8. Find a point in the base BC of a triangle ABC from which the perpendiculars to the two sides AB, AC shall be equal. XV. 8.

9. Find a point equidistant from two given points A and B, and one inch from a given point C. When is the problem impossible? XV. 9.

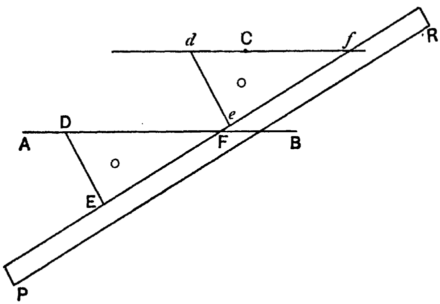
10. In the side AB, or in AB produced, of a triangle ABC, find a point equidistant from B and C. XV. 10.

11. Find a point D in the side BC of a triangle ABC such that AD may be half the sum of AB and AC. XV. 11.

12. ABCD is a quadrilateral; find a point E such that the two straight lines EA, EB shall be equal to the two straight lines ED, EC respectively. XV. 12.

PROPOSITION 26. PROBLEM.

To draw a straight line, through a given point, parallel to a given straight line.



Let AB be the given str. line, and C the given pt.

It is reqd. to draw through C *a str. line* || *to* AB.

Place a set-square DEF with one edge DF in the line AB, and against its edge EF place a flat ruler PR.

Slide the set-square along the ruler until its edge DF passes through the pt. C, the set-square then being in the position *def*.

Trace the line *d*C*f* along its edge.

*d*C*f is the line reqd.*

For the $\angle dfe$ = an $\angle$ of the set-square,

and the $\angle$ DFE = the same $\angle$ of the set-square;

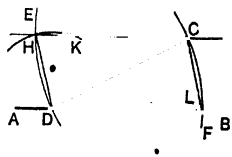
$\therefore$ the ext. $\angle$ DFE = the int. and opp. $\angle dfe$;

$\therefore$ *d*C*f* is $\parallel$ to AB. (I. 5.)

Q.E.F.

PROPOSITION 26. PROBLEM. [ALTERNATIVE METHOD.]

To draw a straight line, through a given point, parallel to a given straight line.



Let AB be the given straight line, and C the given point.

It is reqd. to draw through C *a str. line parallel to* AB.

In AB take any pt. D.

With centre C and rad. CD, describe the circle DE.

With centre D and rad. DC, describe the circle CF, cutting DB at L.

Join CL.

With centre D and rad. equal to CL, describe the circle HK, H being its pt. of intersection with the circle DE on the same side of AB as the pt. C.

Join CH.

CH *will be parallel to* AB.

Join HD.

In the Δs CHD, DLC, (1) CH = DL, *Radii of equal circles.*

(2) CD is common,

(3) HD = CL; *Radii of equal circles.*

$\therefore$ the $\angle$HCD = the $\angle$CDL, (I. 13.)

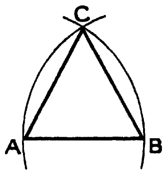
and these are alt. $\angle$s;

$\therefore$ CH is parallel to AB. (I. 4.)

Q.E.F.

PROPOSITION 27. PROBLEM.

To construct an angle of 60°, geometrically.



Draw any str. line AB.

With centre A and rad. AB describe a circle.

" " B " BA " "

Let C be one of the pts. of intersection of these circles. Join CA, CB.

AC = AB ; ∴ ∠C = ∠B.

BA = BC ; ∴ ∠C = ∠A.

∴ the three ∠s of the △ are equal. (I. 11.)

The sum of the three = 180° ;

∴ each of them = 60°.

PROPOSITION 28. PROBLEM.

To construct an angle of 45°, geometrically.

Draw AB perp^r to any str. line BC, using Prop. 22, Bk. I., or Prop. 23, Bk. I., or with ruler and set-square.

On BA and BC mark off, with compasses, BD = BE. Join DE.

Each of the ∠s BDE, BED = 45°.

(The proof is left to the student.)

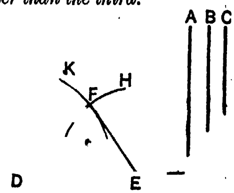
PROPOSITION 29. PROBLEM.

To construct an angle of 30°, geometrically.

Draw an angle of 60° and bisect it, or draw its complement.

PROPOSITION 30. PROBLEM.

To describe a triangle having its sides equal to three given straight lines, any two of which are together greater than the third.



Let A, B, C be the three given str. lines, any two being together gr. than the third.

Take any str. line DE, and with centre D and rad. equal to A describe a circle cutting DE at E.

With centre D and rad. equal to B describe a circle KF.

With centre E and rad. equal to C describe a circle FH.

Take F, one of the pts. of intersection of the circles.

Join DF, EF.

DEF *is the* △ *reqd.*

for DE = A, DF = B, and EF = C. *Cons.*

Q.E.F.

NOTE.—The circles will meet at a second point, so that this construction will give two triangles satisfying the required conditions.

EXERCISES.

1. Draw a triangle whose sides are respectively 5·4, 4·3, 6 cm. long.

2. Draw a triangle whose sides are respectively 4·8, 6·4, 8 cm. long, and measure its greatest angle.

3. On a base 2·3 in. long, draw an isosceles triangle whose other sides are 3·4 in. long.

PROPOSITION 31. PROBLEM.

To construct a triangle having given two sides and the included angle.

If AOB is the given ∠, from OA and OB, with a pair of compasses, cut off OC, OD respectively equal to the given sides. Join CD.

OCD *is the reqd.* △.

PROPOSITION 32. PROBLEM.

To construct a triangle having given one side and the adjacent angles.

If AB is the given side, draw the angles BAC, ABC equal to the given angles, using the method of I. 25.

ABC *is the reqd.* △.

PROPOSITION 33. PROBLEM.

To construct a triangle having given one side and two angles, one angle being opposite the given side.

If AB is the given side, draw the ∠BAC equal to the ∠ which is not opposite to AB.

On the opposite side of AC to the pt. B, draw the ∠CAD equal to the ∠ opp. AB.

Through B draw BE ∥ to AD cutting AC at E.

AEB *is the reqd.* △, for ∠AEB = ∠DAE. *Alt.* ∠*s.*

PROPOSITION 34. THEOREM. [THE AMBIGUOUS CASE.]

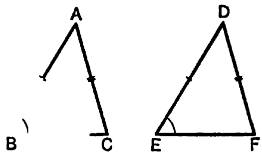
The following proposition is important; it will be seen that in one case the triangles are equal in all respects.

If two triangles have two sides of the one respectively equal to two sides of the other, and also the angles equal which are opposite to one pair of equal sides, then the angles opposite the other pair of equal sides are either equal or supplementary.

Let $\triangle$s ABC, DEF be such that

$AB = DE,\ AC = DF,$

and $\angle ABC = \angle DEF.$



It is reqd. to prove that $\angle$*s* ACB, DFE *are equal or supplementary.*

$\angle BAC$ must be either equal to $\angle EDF$ or not.

(1) When $\angle BAC = \angle EDF$.

In $\triangle$s BAC, EDF, (1) $\angle BAC = \angle EDF$, *Given.*

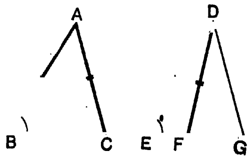
(2) $\angle ABC = \angle DEF$, "

(3) $AB = DE$; "

$\therefore\ \angle ACB = \angle DFE$,

and the $\triangle$s are equal in all respects. (I. 10.)

(2) When $\angle BAC$ is not equal to $\angle EDF$.



At the pt. D, on the same side of DE as the pt. F, suppose the $\angle EDG$ made equal to the $\angle BAC$, and let EF, produced if necessary, meet DG at G.

Then in the $\triangle$s ABC, DEG,

(1) $\angle ABC = \angle DEG$, *Given.*

(2) $\angle BAC = \angle EDG$, *Cons.*

(3) $AB = DE$;

$\therefore\ AC = DG$ and $\angle ACB = \angle DGE$. (I. 12.)

Also $DF = DG$; $\therefore\ \angle DFG = \angle DGF$; (I. 11.)

$\therefore\ \angle DFE =$ the supplement of $\angle DFG$

$=$ the supplement of $\angle DGF$

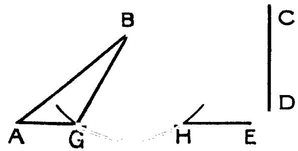
$=$ the supplement of $\angle ACB$.

Wherefore, if two triangles, etc. Q.E.D.

This proposition may be omitted from a first reading.

PROPOSITION 35. PROBLEM.

To construct a triangle having given two sides and the angle opposite one of them. (THE AMBIGUOUS CASE.)

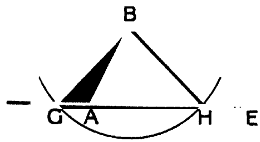


Let AB, CD be the given sides.

If the ∠ opp. CD is given, draw the ∠BAE = the given angle.

With centre B and rad. CD, describe a circle cutting AE at G and H. Join BG, BH.

(*N.B.*—If the circle does not meet AE, the construction is impossible.)



1. If G and H lie on the same side of AB as the ∠BAE, we shall have two △s satisfying the given conditions, viz.

△AGB and △AHB.

2. If G and H fall as in the second figure, we have but one △ (△AHB) satisfying the given conditions.

3. If the circle touches AE, *i.e.* meets it at one point only, we shall again have but one △ as required.

Construction of a triangle from given data.

If three elements or parts of a triangle are given, the triangle can be constructed, provided that one of the given elements is a side.

The different cases are specified below, and are very important.

(1) When the three sides are given.

(2) When two sides and the included angle are given.

(3) When two sides and the angle opposite one of them are given.

The ambiguous case.

(4) When one side and two angles are given.

(The third angle may be found by means of Prop. 7.)

IMPORTANT MISCELLANEOUS EXERCISES.

1. From two given points on the same side of a given straight line, draw two straight lines meeting in the given line, and making equal angles with it. XVIII. 2.

(*Analysis.* Let A and B be the given pts., and CD the given line. *Suppose that* AE *and* EB *are the reqd. lines.* Draw AF perp^r to CD, and let it, when produced, meet BE produced in K. We then see that AF = FK, and the reqd. construction easily follows.)

2. Any two sides of a triangle are together greater than twice the line joining the vertex to the middle point of the base. XVIII. 4.

(If AD be the line from the vertex A to the middle point of the base BC, produce AD to E, making DE equal to AD. Join CE. Prove that CE = AB. The proposition then follows. This is a useful construction.)

3. Construct an isosceles triangle having given the base, and the sum of one of the equal sides and the perpendicular from the vertex to the base. XVIII. 5.

4. Construct a triangle having given the base, one of the angles at the base, and the sum of the sides. XVIII. 6.

(*Analysis.* Let AB be the given base, BAC the given angle, and AC the given sum of the sides. Join BC. *Then if* ADB *be the reqd.* △, we see that BD = DC, and therefore the ∠DBC = the ∠DCB. The reqd. construction now easily follows.)

5. Construct a triangle having given the perimeter and the angles at the base. XVIII. 7.

(*Analysis. Suppose that* ABC *is the reqd.* △. Then if BC be produced both ways so that BD = BA and CE = CA, DE is the perimeter of the △, and is therefore known. Moreover, the ∠ADB = one half the ∠ABC, and the ∠AEC = one half the ∠ACB. Hence since the ∠s ABC, ACB are given, the reqd. construction easily follows.)

6. Find a point in a given straight line so that the sum of its joins to two given points on the same side of the given straight line may be as small as possible. XVIII. 8.

7. The middle point of the hypotenuse of a right-angled triangle is equidistant from all the angular points. XVIII. 9.

(If D be the middle point of the hypotenuse AB, prove that DC cannot be less or greater than DA or DB.)

EXERCISES.

1. Construct a right-angled triangle having given the hypotenuse and one side. XVI. 15.

2. Construct a right-angled triangle having given the hypotenuse and the perimeter. XVI. 16.

3. Construct an equilateral triangle having given the perpendicular from the vertex to the base. XVI. 17.

4. Construct an isosceles triangle having given the vertical angle, and the length of the perpendicular from it to the base. XVI. 13.

5. Construct an isosceles triangle having given its perimeter, and the length of the perpendicular from the vertex to the base. XVI. 14.

MENSURATION EXERCISES.

1. Draw a triangle whose sides are $3\frac{1}{2}$, $2\frac{1}{2}$, and 4 inches long. Measure its sides in centimetres. XVII. 1.

2. Make a triangle whose sides are 4, 5, 6 centimetres long, and measure its sides in inches and decimals of an inch. XVII. 2.

3. Draw a right-angled triangle having its hypotenuse 5 centimetres long, and one of its angles equal to 35 degrees. Explain your construction. XVII. 3.

4. Describe an equilateral four-sided figure. Cut it out in paper, fold it about its diagonals, and deduce facts, giving your reasons. XVII. 4.

5. Construct an isosceles triangle on a base of 2 inches, having each of its base angles equal to 40 degrees. Measure the equal sides. XVII. 5.

6. Find in degrees the angle of a regular pentagon; and on a base of 3 centimetres, with the help of a protractor, make a regular pentagon. XVII. 6.

7. Describe a regular hexagon on a base of 4 centimetres. XVII. 7.

8. On a base of 2 inches describe a triangle with base angles of 30 and 50 degrees. Measure its sides in centimetres. XVII. 8.

9. Make a triangle whose angles are proportional to 1, 2, and 3. XVII. 9.

10. Describe how to draw a perpendicular to a given straight line by folding. XVII. 10.

11. With ruler and compasses (no protractor), make an angle of 30 degrees. Explain your construction. XVII. 11.

12. Make a right-angled triangle having one of the acute angles double the other. Measure the hypotenuse and the shortest side. Prove geometrically that in such a triangle the longest side is double the shortest. XVII. 12.

13. On a base 1·4 inches long, draw an isosceles triangle whose altitude is 2·4 inches. XVII. 13.

14. Two sides of a triangle are respectively 2·5 inches and 3·4 inches in length, and the included angle is 33 degrees. Draw the triangle and measure the third side. XVII. 14.

15. Draw a right-angled triangle whose sides containing the right angle are respectively 3·7 cm. and 3·2 cm. long. Ascertain by measurement the length of the other side. XVII. 15.

16. From a given point C draw a straight line one inch long to a given straight line AB. How many solutions do you obtain, and when is a solution impossible? XVII. 16.

17. On a base of 3 centimetres draw an isosceles triangle having its equal sides together equal to 7 centimetres. XVII. 17.

18. A tower subtends an angle of 45 degrees at a point in the horizontal plane through its foot at a distance of 50 feet from it: find the height of the tower. XVII. 18.

19. A and B are two points on a circle of radius 2 inches. If the straight line AB is 3 inches long, find, with the help of a protractor, the angle which AB subtends at the centre of the circle. XVII. 19.

20. Describe a triangle ABC, such that $AB=3$, $BC=4$, and $CA=5$ centimetres. Measure the length of the line joining the point B to the middle point of AC. XVII. 20.

21. Describe a triangle ABC, such that $AB=3$ inches, the $\angle CAB=40$, and the $\angle CBA=60$ degrees. Measure the sides CA, CB. XVII. 21.

22. Describe a triangle having two of its sides 2 and $2\frac{1}{2}$ inches long, and the included angle equal to 40 degrees. Measure its third side in inches and decimals of an inch. XVII. 22.

23. Draw a line 4 inches long and measure it in centimetres. Hence express 1 inch in centimetres: and explain how it is that a more accurate result can be obtained in this way than by measuring a line 1 inch in length. XVII. 23.

24. A tower subtends an angle of 30 degrees at a point in the horizontal plane through its foot at a distance of 300 feet from it. Taking an inch to represent 100 feet, draw a diagram representing the tower, and measure its height. XVII. 24.

25. A tower has an elevation of 60 degrees from a point A; and from B, 30 feet farther away, the elevation is 45 degrees. Taking a centimetre to represent 10 feet, draw a diagram to represent the tower, and measure its height. XVII. 25.

26. C is a point at equal distances from two points A and B, and the angle ACB = 100 degrees: taking an inch to represent 100 feet, draw an accurate diagram, and measure the length of AC when AB = 300 feet. XVII. 26.

27. A flagstaff stands on a mound 20 feet high. From a point A at the foot of the mound the elevations of the foot and the summit of the flagstaff are 45 and 60 degrees respectively: taking an inch to represent 20 feet, draw a diagram and find the distance of the point A from the top of the flagstaff. XVII. 27.

28. Draw, without the help of a protractor, as accurately as you can, an angle of $22\frac{1}{2}$ degrees. Describe your construction. XVII. 28.

29. Construct a triangle whose base is 5 cm., perimeter 16 cm., and one base angle 62°. XVII. 29.

30. Construct a triangle whose base is 5 cm., altitude 4 cm., and one base angle 45°. XVII. 30.

31. Draw a pavement consisting of equal regular hexagons. Draw one composed of squares and regular octagons. XVII. 31.

32. A and B are fixed points. Draw AC, BC so that the angle C may be 15°. XVII. 32.

33. A triangle has sides 2·5, 3, 3·5 cm. Find a point equidistant from all its vertices. Find also a point equidistant from its sides. XVII. 33.

34. A triangle has sides AB = 4, BC = 5, CA = 4·5 cm. Find a point equidistant from A and B and 2 cm. from C. XVII. 34.

BOOK II.

DEFINITIONS.

1. A **parallelogram** is a four-sided figure which has its opposite sides parallel.

2. A **rectangle** is a parallelogram which has one of its angles a right angle.

(It will be shown hereafter that all the angles of a rectangle are right angles.)

3. A **square** is a rectangle which has two adjacent sides equal.

(It will be proved that all the sides of a square are equal, and all its angles right angles.)

4. A **trapezium** is a four-sided figure which has two of its sides parallel.

5. The **altitude** of a parallelogram is the perpendicular distance between the base and its opposite side.

6. The **altitude** of a triangle is the perpendicular distance of the vertex from the base.

(Thus a triangle may be said to have three altitudes, one drawn from each angular point to the opposite side.)

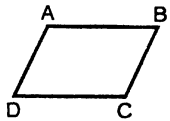
7. The straight line joining any angular point of a triangle to the middle point of the opposite side is called a **median** of the triangle.

8. A **rhombus** is a four-sided figure, which has all its sides equal, but its angles are not right angles.

PROPOSITION 1. THEOREM.

The opposite angles of a parallelogram are equal; and conversely, if the opposite angles of a quadrilateral are equal, the figure is a parallelogram.

Let ABCD be a parallelogram.



It is reqd. to prove that $\angle A = \angle C$ and $\angle B = \angle D$.

Since AD is $\parallel$ to BC, $\angle A$ is the supplement of $\angle B$. (I. 5.)
Since AB is $\parallel$ to DC, $\angle C$ is the supplement of $\angle B$. (I. 5.)

$$\therefore \angle A = \angle C.$$

Similarly $\angle B = \angle D$.
Conversely, let $\angle A = \angle C$ and $\angle B = \angle D$.
It is reqd. to prove ABCD *a parallelogram.*
Here $\angle A + \angle B = \angle C + \angle D = \frac{1}{2}$ the sum of the four $\angle$s.
But the sum of the 4 angles = 4 rt. $\angle$s, since the figure may be divided into two triangles. (I. 7.)

$$\therefore \angle A + \angle B = 2 \text{ rt. } \angle\text{s};$$

$$\therefore \text{AD is} \parallel \text{to BC.}$$

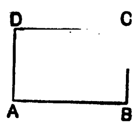
Also $\angle A + \angle D = \angle A + \angle B = 2$ rt. $\angle$s;

$$\therefore \text{AB is} \parallel \text{to DC} \qquad \text{(I. 4.)}$$

Q.E.D.

COROLLARY. *The angles of a rectangle are all right angles.*

Let ABCD be a rectangle having the $\angle$BAD a rt. angle.

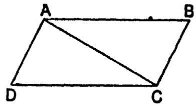


Since the fig. is a parallelogram,
the $\angle$BCD = the opp. $\angle$BAD = a rt. $\angle$.
Also, since DC is parallel to AB,
the two int. $\angle$s CDA, DAB are together equal to two rt. $\angle$s.
But the $\angle$BAD is a rt. $\angle$. *Given.*
$\therefore$ the $\angle$CDA is a rt. $\angle$;
$\therefore$ the opp. $\angle$CBA is a rt. $\angle$. Q.E.D.

Proposition 2. Theorem.

The opposite sides of a parallelogram are equal to one another, and each diagonal bisects the parallelogram; and the converse.

Let ABCD be a par^m, and AC a diagonal.



It is reqd. to prove that

(1) AD = BC, (2) AB = CD,

(3) the △ADC = the △CBA.

In the △s ADC, CBA,

(1) the ∠DAC = the ∠BCA, *alt.* ∠*s*, (I. 5.)

(2) the ∠DCA = the ∠BAC ,,

(3) AC is common;

∴ AD = BC,

CD = AB,

and the △ADC = the △ABC in area. (I. 10.)

Q.E.D.

Conversely, if the opposite sides of a quadrilateral are equal, the figure is a parallelogram.

In the quadrilateral ABCD let AB = CD, and AD = BC.

Prove that ABCD *is a parallelogram.*

In the △s ABC, CDA, (1) AB = CD, *Given.*

(2) BC = DA, ,,

(3) AC is common;

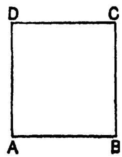
∴ ∠ACB = ∠CAD and ∠CAB = ∠ACD. (I. 13.)

Since ∠ACB = alt. ∠CAD, BC is ∥ to AD. (I. 4.)

Since ∠CAB = alt. ∠ACD, AB is ∥ to DC. (I. 4.)

Q.E.D.

Corollary. *All the sides of a square are equal, and all its angles are right angles.*



Let ABCD be a square.

Since the fig. is a rectangle, *Def.*

its angles are all right angles. *Cor.* 1.

Also since the fig. is a parallelogram, *Def.*

its opposite sides are equal. (II. 2.)

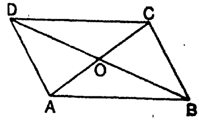
But, by def., two of its adj. sides are equal;

∴ all its sides are equal.

Q.E.D.

PROPOSITION 3. THEOREM.

The diagonals of a parallelogram bisect one another, and the converse.



Let the diagonals AC, BD of the parallelogram ABCD intersect at O.

Then in the Δs AOD, COB,

(1) the vertically opp. $\angle$s AOD, COB are equal, (I. 2.)

(2) the alt. $\angle$s ADO, CBO are equal, (I. 5.)

(3) AD = BC; (II. 2.)

$\therefore$ AO = CO, and BO = DO. (I. 10.)

Q.E.D.

Conversely, if the diagonals of a quadrilateral bisect each other, the quadrilateral is a parallelogram.

Let ABCD be a quadrilateral whose diagonals bisect each other at O.

It is reqd. to prove that ABCD *is a parallolegram.*

In the Δs AOD, COB,

(1) AO = CO, *Given.*

(2) DO = BO, „

(3) $\angle$AOD = $\angle$COB (*vert. opp.*); (I. 3.)

$\therefore$ AD = CB, and $\angle$ADO = $\angle$CBO. (I. 9.)

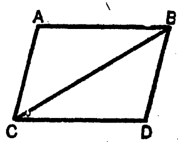
Since $\angle$ADO = alt. $\angle$CBO,

AD is || to CB. (I. 4.)

Similarly, from the Δs AOB, COD, AB is || to CD. Q.E.D.

PROPOSITION 4. THEOREM.

The straight lines which join the extremities of equal and parallel straight lines towards the same parts are themselves equal and parallel.



Let AB, CD be two equal and || str. lines joined towards the same parts by the str. lines AC, BD.

It is reqd. to prove that AC *is equal and* || *to* BD.

Join BC.

$\because$ AB is || to CD; $\therefore$ $\angle$ABC = alt. $\angle$BCD. (I. 5.)

In Δs ABC, DCB,

(1) AB = CD, *Given.*

(2) BC is common,

(3) included $\angle$ ABC = included $\angle$ BCD ; *Proved above.*

$\therefore$ AC = BD, and $\angle$BCA = $\angle$CBD, (I. 9.)

but these are alternate $\angle$s ;

$\therefore$ AC is $\parallel$ to BD. (I. 4.)

Q.E.D.

EXERCISES.

1. Describe a parallelogram having one side 1 inch and its diagonals 2 and 1·6 inches long.

2. Bisect a straight line by means of a ruler and a set-square. XIX. 4.

3. Every rhombus is a parallelogram. XIX. 6.

4. **The diagonals of a rhombus bisect one another at right angles.** XIX. 10.

5. Draw a straight line parallel to the base BC of a triangle ABC such that the part of it intercepted between the two sides AB, AC may be equal to 2 cm., BC being greater than 2 cm. XIX. 11.

6. ABCD, ABEF are two parallelograms having a common side AB. If CE, DF be joined, show that CDFE is a parallelogram. XIX. 12.

7. The extremities of two diameters of a circle are joined; prove that the figure thus formed is a rectangle. XIX. 14.

8. E, F, H, K are points in the sides AB, BC, CD, DA respectively of a parallelogram ABCD, such that AK is equal to FC, and AE is equal to CH. Show that EFHK is a parallelogram. XIX. 15.

9. If two opposite sides of a quadrilateral are parallel, and the other two sides are equal and not parallel, prove that the angles adjacent to either of the first pair of sides are equal. XIX. 16.

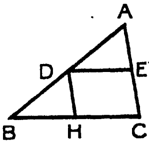
10. **Through a given point D within a given angle BAC, draw a straight line BDC so that BD is equal to DC.** XIX. 20.

11. **Every straight line drawn through the intersection of the diagonals of the parallelogram divides the parallelogram into two equal parts.** XIX. 22.

12. The longer sides of a parallelogram are twice as long as the shorter sides. Show that the straight lines joining the middle point of one of the longer sides with the ends of the opposite side, are perpendicular to one another. XIX. 13.

PROPOSITION 5. THEOREM.

The straight line drawn through the middle point of one side of a triangle parallel to another side bisects the third side.



Let ABC be a △, D the mid-point of AB, and DE ∥ to BC.

It is reqd. to prove that DE *bisects* AC.

From D draw DH ∥ to AC to meet BC at H.

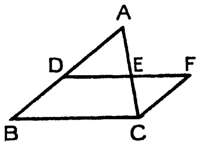
In the △s DAE, BDH,

(1) ∠DAE = ∠BDH, since AE is ∥ to DH.	(I. 5.)
(2) ∠ADE = ∠DBH, since DE is ∥ to BH.	"
(3) AD = DB;	*Given.*
∴ AE = DH	(I. 10.)
= EC.	*Opp. sides of a parm.*

Q.E.D.

PROPOSITION 6. THEOREM.

The straight line joining the middle points of two sides of a triangle is parallel to the third side, and equal to one-half of it.



Let ABC be a triangle, and D, E the mid-points of AB, AC.

It is reqd. to prove that DE *is ∥ to* BC, *and equal to one-half of* BC.

Produce DE to F so that EF = DE.

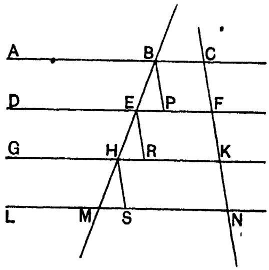
Join FC.

In the △s AED, CEF,

(1) DE = EF,	*Cons.*
(2) AE = EC,	*Given.*
(3) ∠AED = ∠CEF (*vert. opp.*);	(I. 3.)
∴ AD = FC, and ∠ADE = ∠EFC.	(I. 9.)
Since ∠ADE = alt. ∠EFC, AB is ∥ to FC.	(I. 4.)
But FC = AD	
= DB.	*Given.*
Since FC is equal and ∥ to DB,	
DF is equal and ∥ to BC.	(II. 4.)
Also DE is half of DF;	
∴ DE is half of BC.	Q.E.D.

PROPOSITION 7. THEOREM.

If several parallel straight lines cut equal intercepts on one transversal, they do so on all transversals.



Let a transversal (*i.e.* any intersecting line) BEHM cut the parallel str. lines AC, DF, GK, LN so that the intercepts BE, EH, HM are equal to one another.

Let CFKN be *any* other transversal, as shown in the fig.

It is reqd. to prove that

CF = FK = KN.

Draw BP, ER, HS parallel to CFKN as shown in the fig.

In the $\triangle$s BEP, EHR,

(1) BE = EH, *Given.*

(2) ext. $\angle$BEP = int. opp. $\angle$EHR, (I. 5.)

(3) $\angle$EBP = $\angle$HER; (I. 5.)

$\therefore$ BP = ER. (I. 10.)

But BF and EK are parallelograms;

$\therefore$ CF = BP and FK = ER; (II. 2.)

$\therefore$ CF = FK.

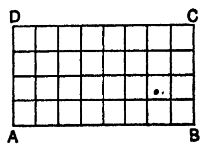
In the same way it may be proved that FK = KN, and so on.

Q.E.D.

Before learning the next proposition, the student should work some examples on the use of squared paper.

PROPOSITION 8. THEOREM.

The area of a rectangle is measured by the product of the measures of its length and breadth.



Let ABCD be a rect. having AB a units and AD b units of length.

Let AB be divided into equal parts, each one unit in length; so that there are a parts.

Also let AD be divided into equal parts, each one unit in length; so that there are b parts.

Through the pts. of division draw parallels to AB and AD respectively. The rect. is thus divided into a number of sqs. each one unit long and one unit broad.

Moreover we see that there are b rows, each containing a squares. ∴ the area of the rect. is equal to ab unit sqs.

Thus if we take an *inch* as the unit of length, the area of the rect. will be ab sq. inches.

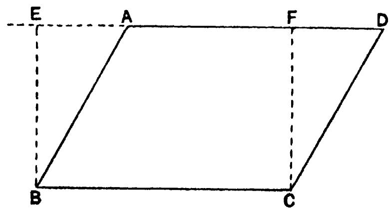
The above result may be stated as follows: "**The number of units of length in the length, multiplied by the number in the breadth, will give the number expressing the area of a rectangle.**"

In a condensed form it is: "*The area of a rectangle is the product of the length and the breadth.*"

The proof holds for the case in which the lengths of the sides are not whole numbers. For instance, let the sides of a rectangle be 2·7 in. and 3·2 in. Take one-tenth of an inch for unit of length. Then the sides contain 27 units and 32 units respectively, and the argument applies equally to this case.

Proposition 9. Theorem.

The area of a parallelogram is equal to the area of a rectangle on the same base and between the same parallels, and is measured by the product of the measures of its base and altitude.



Let ABCD be any parallelogram.

Let BE, CF be perpendiculars drawn from B, C to AD.

In the right-angled △s CFD, BEA,

CD = BA (opp. sides of a parallelogram)

and CF = BE ,, ,, ,,

∴ the △CFD = the △BEA. (I. 14.)

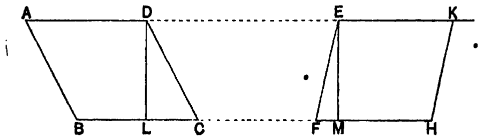
Add to each the figure ABCF.

Then the par^m ABCD = the rectangle EBCF

= EB . BC = the product of the altitude and base. Q.E.D.

Proposition 10. Theorem.

Parallelograms of equal altitudes on equal bases are equal in area.



Let ABCD, EFHK be parallelograms of which the base BC = the base FH, and the altitude DL = the altitude EM.

It is required to prove that the parallelograms are equal in area.

The par^{m} ABCD = DL . BC

= EM . FH *Given.*

= the par^{m} EFHK.

This proposition includes both the following:

Parallelograms on the same base and between the same parallels are equal in area; parallelograms on equal bases and between the same parallels are equal in area.

Proposition 11. Theorem.

The area of a triangle is equal to one-half of the area of a rectangle on the same base and between the same parallels.

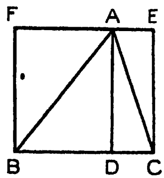


Fig. 1.

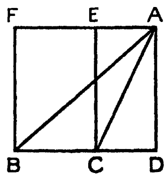


Fig. 2.

Let the $\triangle$ABC and the rect. BCEF be on the same base BC, and between the same ||s BC and EF.

It is reqd. to prove that $\triangle$ABC = $\frac{1}{2}$ *rect.* BCEF.

Draw AD perp^{r} to the base BC, produced if necessary.

In Fig. 1, $\triangle$ABC = $\triangle$ABD + $\triangle$ADC

= $\frac{1}{2}$ rect. FD + $\frac{1}{2}$ rect. DE (II. 2.)

= $\frac{1}{2}$ rect. FC.

In Fig. 2, $\triangle ABC = \triangle ABD - \triangle ADC$

$= \frac{1}{2}$ rect. FD $- \frac{1}{2}$ rect. DE (II. 2.)

$= \frac{1}{2}$ rect. FC. Q.E.D.

COR. 1. *The area of a triangle is measured by one-half the product of the measures of its base and altitude.*

By the above proposition,

$$\triangle ABC = \tfrac{1}{2} \text{ rect. } FC = \tfrac{1}{2} . BF . BC = \tfrac{1}{2} AD . BC.$$

COR. 2. From Cor. 1 it follows that:

Triangles on the same or equal bases and of the same altitude are equal in area.

EXERCISES.

1. Find the area of a $\triangle$ 2 ft. high. on a base of 14 inches.

2. Draw a $\triangle$ with sides 3, 4, 5 cm. Given that it is right-angled, find its area. Calculate the length of the altitude drawn to the longest side.

3. In a right-angled $\triangle$ whose sides are 5, 12, 13 cm., calculate the length of the perp[r] drawn from the rt. $\angle$ to the opposite side.

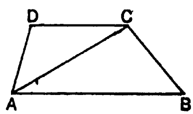
4. In a right-angled $\triangle$ whose sides are of length a, b, c, prove that $ab \div c$ is the length of the altitude drawn to the hypotenuse c.

5. Find, by measurement of the altitude, the area of an equilateral $\triangle$ on a base of 4 inches. XXIV. 11.

PROPOSITION 12. THEOREM.

The area of a trapezium is half the product of the sum of the parallel sides and the perpendicular distance between them.

Let ABCD be a trapezium, having AB par[l] to CD. Join AC. Let AB be a units, CD b units, and the common altitude of the $\triangle$s ABC, DAC h units in length.



(The altitude of each $\triangle$ is the perp[r] distance between the parallels AB, CD.)

Then the area of ABCD

$= $ the $\triangle ABC +$ the $\triangle ACD$

$= \frac{1}{2}ah + \frac{1}{2}bh$

$= \frac{1}{2}h\,(a + b)$.

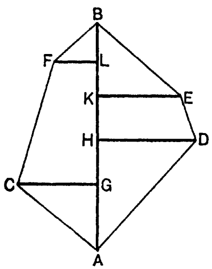
THE FIELD-BOOK.

In surveyors' work the area of a rectilinear plane figure is found by simply dividing it up into right-angled triangles and trapeziums.

The division is performed by joining two angular points and by measuring the perpendicular distances of the other angular points from this **base line.** The perpendiculars are called **offsets.**

The method will be understood from the following plan of a field and its record in the **field-book.** The field-book is to be read upwards. The numbers in the middle column show the distances of G, H, K, L, B respectively *all measured from* A; the other columns show the lengths of the perpendicular offsets left

Plan.



Field-book.

	Links	
	To B	
	279	
to F 68	230	
	190	110 *to* E
	130	128 *to* D
to C 110	80	
	From A	

and right. Thus CG = 110, HD = 128, KE = 110, FL = 68. It only remains to calculate the areas of the four right-angled triangles and the two trapeziums.

EXERCISES.

Draw plans and find areas of fields from the following entries in the Field-book:

1. Links.

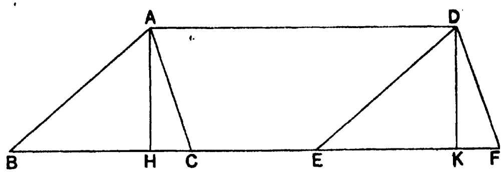
128	960	
	896	224
	768	240
320	752	
	608	160
240	160	

2. Links.

	360	
50	300	
	244	90
90	180	
	144	80
	0	20

PROPOSITION 13. THEOREM.

Triangles equal in area on equal bases in the same straight line and on the same side of that straight line are between the same parallels.



Let ABC, DEF be two equal $\triangle$s on equal bases BC, EF.

Let BC, EF be in one str. line, and let the $\triangle$s be on the same side of it.

It is reqd. to prove that AD *is* || *to* BF.

Let AH, DK be perp^r to BF.

The $\triangle ABC = \frac{1}{2} AH \cdot BC$,

and the $\triangle DEF = \frac{1}{2} DK \cdot EF$.

But the $\triangle ABC =$ the $\triangle DEF$; *Given.*

$\therefore AH \cdot BC = DK \cdot EF$.

But $BC = EF$; *Given.*

$\therefore AH = DK$.

Moreover, AH is || to DK, since both are perp^r to HK;

$\therefore$ AD is || to HG. Q.E.D.

The same proof would apply to the proposition—*Triangles equal in area on the same base and on the same side of it are between the same parallels.*

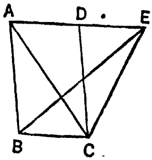
IMPORTANT EXERCISES.

1. The joins of the middle points of the sides of a triangle divide the triangle into four congruent triangles.

2. In any quadrilateral, (1) the straight line joining the middle points of two adjacent sides is parallel to a diagonal and equal to half of it; (2) the figure formed by joining the middle points of the sides in order is a parallelogram; (3) the straight lines joining the middle points of opposite sides bisect one another. XX. 2.

PROPOSITION 14. THEOREM.

If a triangle and a parallelogram be on the same base and between the same parallels, the parallelogram is double of the triangle.



Let ABCD be a par^{m} and EBC a $\triangle$ on the same base and between the same ||s ADE and BC.

It is reqd. to prove that the parallelogram is double of the triangle.

Join AC.

AC is the diagonal of the par^{m};

$\therefore$ the $\triangle$ADC = the $\triangle$ABC.

$\therefore$ the par^{m} ABCD is double of the $\triangle$ABC.

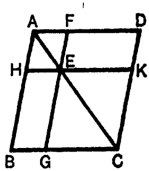
But the $\triangle$ABC = the $\triangle$EBC (on the same base and of equal altitudes);

$\therefore$ the par^{m} ABCD is double of the $\triangle$EBC. Q.E.D.

DEFINITION.—If through a point in the diagonal AC of a parallelogram straight lines be drawn parallel to the sides, the figure is divided into four parallelograms. Of these, the two whose diagonals lie in AC are called **parallelograms about the diagonal** AC, and the other two are called the **complements** of these.

PROPOSITION 15. THEOREM.

The complements of the parallelograms about the diagonal of any parallelogram are equal.



Let ABCD be a par^{m}, and BE, DE the complements of the par^{ms} about the diagonal AC formed by drawing FEG, HEK || to AB and AD respectively.

It is reqd. to prove that the complement BE = *the complement* DE.

EGCK is a par^{m} and EC a diagonal; *Cons.*

$\therefore$ $\triangle$EGC = $\triangle$EKC. (II. 2.)

In the same way, $\triangle$AHE = $\triangle$AFE;

$\therefore$ $\triangle$s EGC, AHE = $\triangle$s EKC, AFE.

But the whole $\triangle$ABC = the whole $\triangle$ADC; (II. 2.)

$\therefore$ **the remainder, the complement** BE,

= **the remainder, the complement** DE. Q.E.D.

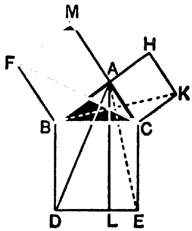
EXERCISE.

Given a rectangle 6 cm. by 3 cm., construct one of equal area which shall have one side 4·5 cm. in length.

(Construct a rectangle so that these two rectangles shall be the complements of the rectangles about the diagonal.)

PROPOSITION 16. THEOREM OF PYTHAGORAS.

In any right-angled triangle, the square described on the side subtending the right angle is equal to the squares described on the sides containing the right angle.



Let ABC be a rt.-angled △, having the rt. ∠ at A.

It is reqd. to prove that the sq. on BC = *the sqs. on* AB *and* AC.

Let BDEC be the sq. on BC, and ABFM, ACKH the sqs. on AB, AC respectively.

Also let AL, drawn ∥ to BD or EC, meet DE at L.

Join AD, CF;

the ∠s BAM, BAC are rt. ∠s;

∴ MA and AC are in one str. line; (I. 2.)

∴ the sq. BM = twice the △FBC (on the same base and between the same parallels). (II. 11.)

For the same reason the rect. BL = twice the △ ABD.

Now the ∠s FBA, DBC are rt. ∠s;

∴ adding the ∠ABC to each, the ∠FBC = the ∠ABD.

Then in the △s FBC, ABD

(1) FB = AB *Sides of a sq.*

(2) BC = BD *Sides of a sq.*

(3) the ∠FBC = the ∠ABD; *Proved above.*

∴ the △FBC = the △ABD. (I. 9.)

But the sq. BM = twice the △FBC,

and the rect. BL = twice the △ABD;

∴ the sq. BM = the rect. BL.

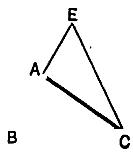
In the same way, by joining BK and AE, it may be shown that the sq. CH = the rect. CL;

∴ the two rects. BL and CL = the sqs. BM and CH,

i.e. the sq. BE = the sqs. BM and CH. Q.E.D.

PROPOSITION 17. THEOREM.

If the square described on one of the sides of a triangle be equal to the squares described on the other two sides, the angle contained by those two sides is a right angle.



Let ABC be a $\triangle$ such that

the sq. on BC = the sqs. on BA and AC.

It is reqd. to prove that the $\angle$BAC *is a right* $\angle$.

Let AE be drawn at rt. $\angle$s to CA, and cut off AE equal to AB.

Join EC.

The $\angle$CAE is a rt. $\angle$.

$\therefore$ the sq. on CE

= the sqs. on CA and AE (II. 16.)

= the sqs. on CA and AB (for AB = AE)

= the sq. on BC; *Given.*

$\therefore$ CE = CB.

Hence in the $\triangle$s BAC, EAC,

(1) AB = AE, *Cons.*

(2) CA is common,

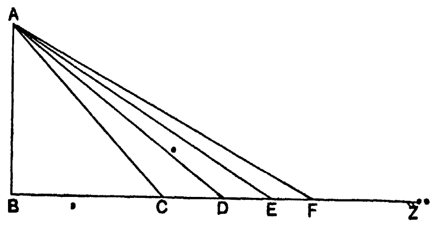
(3) BC = CE; *Proved.*

$\therefore$ the $\angle$BAC = the $\angle$EAC (I. 13.)

= a rt. $\angle$. *Cons.*

Q.E.D.

Geometrical representation of $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, etc.



Draw AB 1 inch in length. Perpendicular to this draw BZ of indefinite length.

Along BZ mark off BC equal to BA. Then AC represents $\sqrt{2}$.

" " BD " AC. Then AD " $\sqrt{3}$.

" " BE " twice AB.

Then AE represents $\sqrt{5}$.

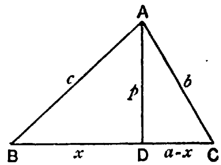
Along BZ mark off BF equal to AE. Then AF represents $\sqrt{6}$, and so on.

Prop. 16 *is very useful in finding the area of a triangle; for it generally enables us to find the altitude.*

If a, b, c denote the sides of a triangle, and $2s = a + b + c$, its area

$$=\sqrt{s(s-a)(s-b)(s-c)}.$$

Draw AD perp^r to BC, and let BD $= x$, so that CD $= a - x$; also let AD $= p$.



$$c^2 = p^2 + x^2; \quad \text{(II. 16.)}$$

$$\therefore\ p^2 = c^2 - x^2.$$

Also $$b^2 = p^2 + (a - x)^2;$$

$$\therefore\ p^2 = b^2 - (a - x)^2;$$

$$\therefore\ c^2 - x^2 = b^2 - (a - x)^2,$$

whence $$x = \frac{c^2 + a^2 - b^2}{2a}.$$

Hence $p^2 = c^2 - x^2 = c^2 - \left(\frac{c^2 + a^2 - b^2}{2a}\right)^2$

$$= \left[c + \frac{c^2 + a^2 - b^2}{2a}\right]\left[c - \frac{c^2 + a^2 - b^2}{2a}\right]$$

$$= \left[\frac{(c+a)^2 - b^2}{2a}\right]\left[\frac{b^2 - (c-a)^2}{2a}\right]$$

$$= \frac{(c+a+b)(c+a-b)(b+c-a)(b+a-c)}{4a^2}.$$

But $c + a + b = 2s$; $\therefore\ c + a - b = 2s - b - b = 2(s - b)$, and similarly,

$$b + c - a = 2(s - a) \quad \text{and} \quad b + a - c = 2(s - c);$$

$$\therefore\ p^2 = \frac{4s(s-a)(s-b)(s-c)}{a^2};$$

$$\therefore\ \frac{ap}{2} = \sqrt{s(s-a)(s-b)(s-c)}.$$

But the area of the $\triangle = \frac{ap}{2}$;

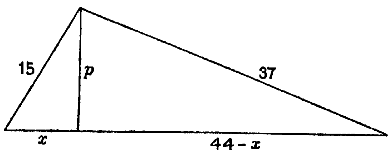
$$\therefore \text{ the area of the } \triangle = \sqrt{s.(s-a)(s-b)(s-c)}. \qquad \text{Q.E.D.}$$

Ex. 1. *Find the area of an isosceles $\triangle$ whose sides are* 17, 17, 16 *ft. The altitude p bisects the base* 16.

$$\therefore \ p = \sqrt{17^2 - 8^2} = \sqrt{(17+8)(17-8)} = \sqrt{25 \times 9} = 5 \times 3 = 15;$$

$$\therefore \text{ the area} = \tfrac{1}{2} \times 16 \times 15 = 120 \text{ sq. ft.}$$

Ex. 2. *Find the area of a $\triangle$ whose sides are* 15, 37, 44 *cm.*
Let the altitude p divide the base 44 *into parts x and* $44 - x$.



$$p^2 + (44 - x)^2 = 37^2,$$
$$p^2 + x^2 = 15^2;$$

$\therefore$ by subtraction

$$44^2 - 88x = 37^2 - 15^2;$$
$$\therefore \ 44(44 - 2x) = (37 + 15)(37 - 15)$$
$$= 52 \times 22$$
$$= 26 \times 44;$$
$$\therefore \ 44 - 2x = 26;$$
$$\therefore \ x = 9.$$
$$p = \sqrt{15^2 - 9^2} = 12;$$
$$\therefore \text{ the area} = \tfrac{1}{2} \times 12 \times 44 = 264 \text{ sq. cm.}$$

EXERCISES.

Find the areas of the triangle whose sides are as follows:

1. 25, 25, 30 cm. **2.** 13, 13, 10 ft. **3.** 13, 14, 15 yds.

4. 39, 41, 50 m. **5.** 25, 52, 63 in. **6.** 12·5, 26, 31·5 m.

NOTES ON II. 16.

(*a*) Pythagoras, a Greek philosopher of the sixth century B.C., is by tradition credited with the earliest discovery of a proof of this proposition, which is consequently often called the Theorem of Pythagoras.

(*b*) In Prop. 16 let AN be perp^r to BC. In the course of the proof it has been shown that the square on AB = the rectangle contained by BN, BC;— a result of great importance.

In the same way, the sq. on AC = the rect. CN . CB.

Also $$AN^2 = BA^2 - BN^2$$
$$= BN . BC - BN^2$$
$$= BN (BC - BN) = BN . NC.$$

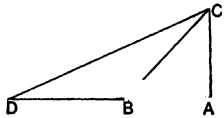
(*c*) To construct a square which shall be twice a given square. Draw the diagonal AC of the given square ABCD. The sq. on AC = twice the square on AB (II. 16).

If CE be drawn at rt. $\angle$s to AC and equal to AB,

then sq. on AE = sq. on AC + sq. on CE (II. 16.)
= three times sq. on AB.

This process may, of course, be continued as far as required.

(*d*) *To produce a given straight line so that the square on the produced part may be twice the square on the given line.*



Let AB be the given straight line.

At right angles to AB draw AC equal to AB.

Join BC.

Produce AB to D making BD equal to BC.

Then the thing required is done.

(*e*) Suppose that we had *to divide a given straight line* into two parts so that the square on one part should be twice the square on the other.

In the foregoing figure AD is a line so divided.

$\therefore$ the question would be simply solved by copying the angles of this figure; *i.e.* by making a right angle at one end of the given line and one-fourth of a right angle at the other end: for by I. 7.

$$\text{twice } \angle D = \angle ABC$$
$$= \text{half a right angle.}$$

(*f*) It may be noticed that $3^2 + 4^2 = 5^2$.

$\therefore$ three straight lines of length 3, 4, 5 respectively will form a right-angled triangle.

The same of course holds for any three numbers which are the same multiple of 3, 4, 5.

5, 12, 13 have the same property.

An unlimited number of groups of 3 numbers having this property may be formed.

The following is an interesting example:

If we take the numbers $1\frac{1}{3}$, $2\frac{2}{5}$, $3\frac{3}{7}$, $4\frac{4}{9}$, etc., and convert them into improper fractions,

the numerators are	4, 12, 24, 40, ...,
and the denominators	3, 5, 7, 9, ...;
the numerators increased by 1,	5, 13, 25, 41, ...

The vertical columns thus obtained give groups of numbers satisfying II. 17.

EXERCISES.

1. Find the length of the hypotenuse of a right-angled triangle whose other sides are 9 in. and 12 in.

2. The hypotenuse of a right-angled triangle is 52 in. long, and one side is 48 inches long. Find the length of the other side.

3. A rt.-angled $\triangle$ has its hypotenuse 5 in. and one side 4 in. long. Determine, by geometry, the third side, and write down the area of the $\triangle$, stating how you arrive at your result. XXIV. 12.

4. If AD be drawn perpendicular to the base BC of a triangle ABC, prove that the difference of the squares on the sides AB, AC is equal to the difference of the squares on the segments BD, CD of the base. XXII. 1.

5. From the vertex of an equilateral triangle a perpendicular is let fall on the base. From II. 16 deduce the length of the perpendicular in terms of the side of the triangle. XXII. 2.

6. On the sides AC, BC of a triangle ABC squares ACDE, BCFH are described outside the triangle; show that the straight lines AF and BD are equal. XXII. 4.

7. Show how to construct a square which shall contain 13 square inches. XXII. 5.

8. Prove that lines of lengths 6, 8, 10 will form a right-angled triangle. XXII. 16.

9. Prove that lines of length 15, 36, 39 will form a right-angled triangle. XXII. 17.

10. Prove that lines of length 28, 96, 100 will form a right-angled triangle. XXII. 18.

11. Draw a rhombus whose sides are 5 cm. long and whose altitude is 4 cm. State exactly how you do the problem. XXIV. 17.

12. Draw a line $\sqrt{6}$ inches long. Measure it carefully.

13. If the sides AB, BC, CD, DA of a square ABCD be produced to P, R, S, T respectively, and the parts produced be each equal to a side of the square, prove that PRST is a square whose area is 5 times that of ABCD. XXII. 11.

Find the areas of the triangles whose sides are as follows:

14. 25, 25, 30 cm. **15.** 13, 13, 10 ft. **16.** 13, 14, 15 yd.

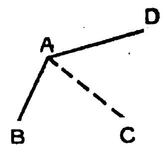
17. 39, 41, 50 m. **18.** 25, 52, 63 in. **19.** 12·5, 26, 31·5 m.

20. In a $\triangle$ whose sides have lengths a, b, c, prove that the altitude, drawn to the base a, cuts off from it one segment equal to $(a^2+b^2-c^2)/2a$.

CONSTRUCTION OF QUADRILATERALS.

Consider any quad[l] ABCD. Join AC.

If it can be drawn from given angles and sides, we must be able to construct the $\triangle$ABC. This can only be done if we know three of the elements or parts of the $\triangle$, one of which must be a side. Also in this case one of the given parts must be an angle, unless AC is given.



The $\triangle$ABC having been drawn, we have to construct the $\triangle$DAC. One part, AC, is now known. Hence we can draw this $\triangle$ if we are given two more of its parts.

Thus we see that in order to draw a quad[l] from given sides and angles we must know five of its parts, and that two of these parts must be sides.

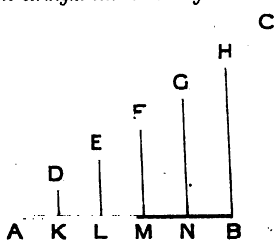
EXERCISES.

QUADRILATERALS.

1. Draw a quadrilateral ABCD rt.-angled at A and B, and such that AB = BC = 3 cm. and CD = 6 cm. in length.
2. Draw the quad[l] ABCD having given AB = AD = 6 cm., BC = 3 cm., $\angle$B = 120°, and AD parallel to BC.
3. Draw the quad[l] ABCD having given AB = BC = 7 cm., DA = DC = 5 cm., and the diagonal AC = 6 cm.
4. Draw a rhombus whose diagonals are 8·6 cm. and 9·4 cm. in length. (See Example 4, p. 65.)

PROPOSITION 18. PROBLEM.

To divide a finite straight line into any number of equal parts.



It is required to divide the str. line AB *into any number of equal parts. (Say five.)*

Draw any str. line AC at an angle with AB, and on it, with a pair of compasses, mark off equal distances AD, DE, EF, FG, GH.

Join HB.

From D, E, F, G draw parallels to HB, cutting AB at K, L, M, N.

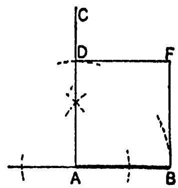
The ||s DK, EL, etc., cut equal intercepts on the transversal AC.

∴ they also cut equal intercepts AK, KL, LM, MN, NB on AB.

Q.E.F.

PROPOSITION 19. PROBLEM.

Describe a square on a given straight line.



Let AB be the given str. line.

It is required to describe a sq. on AB.

From A draw AC at rt. ∠s to AB. (I. 22.)

From AC cut off AD equal to AB.

Through D draw DF || to AB, (I. 26.)

and through B draw BF || to AD meeting DF at F. (I. 26.)

ABFD will be the required sq.

For, by construction, it is a par^m with one of its ∠s a rt. ∠.

∴ it is a rectangle. *Def.*

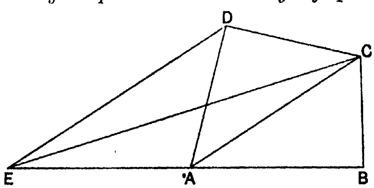
Also AB. AD two adj. sides are equal;

∴ it is a sq., *Def.*

and it is described on AB. Q.E.F.

PROPOSITION 20. PROBLEM.

To reduce a given quadrilateral to a triangle of equal area.



Let ABCD be a quad^l. Join AC.

Through D draw DE || to CA to meet BA produced in E.

Join EC. ECB will be the △ reqd.

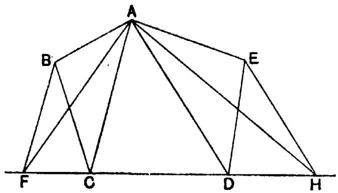
The △ACE = the △ADC, for they are on the same base and between the same parallels.

Add the △ABC to each.

Then the △ECB = the quad^l ABCD in area. Q.E.F.

Proposition 21. Problem.

To reduce a given pentagon to a triangle of equal area.



Let ABCDE be the given pentagon. Join AC.

Through B draw BF ∥ to AC to meet DC produced in F. Join FA.

The △ABC = the △AFC; for they are on the same base and between the same parallels.

Add the figure ACDE to each.

Then the pentagon ABCDE = the quad[l] AFDE.

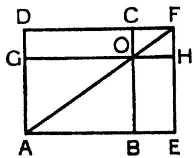
By joining AD, and through E drawing EH ∥ to AD to meet FD produced in H, we can reduce the quad[l] AFDE to an equal triangle AFH.

Thus the given pentagon is reduced to a △ of equal area.

By repeated applications of the same process any polygon can be reduced to a triangle of equal area.

Proposition 22. Problem.

To construct a rectangle equal in area to a given rectangle and having one side of given length.



Let ABCD be the given rect. Along AB cut off AE equal to the given side of the reqd. rect.

Complete the rect. DAEF. Join AF, meeting CB at O.

Draw GOH ∥ to AB or DC, meeting AD at G and EF at H.

GAEH *will be the reqd. rect.*

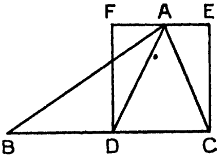
Proof. The complement OE = the complement OD. (II. 15.)

Add the rect. GB to each.

Then Rect. GE = rect. DB. Q.E.F.

PROPOSITION 23. PROBLEM.

To construct a rectangle equal in area to a given polygon and having one side of given length.



Reduce the polygon to a triangle of equal area. (II. 21.)

If ABC is the △, bisect BC at D.

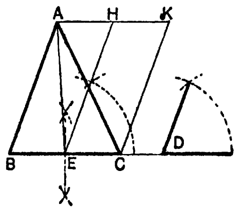
Draw DF and CE perp^r to BC, and EAF ‖ to DC. Join AD.

Rect. FDCE = twice △ADC = △ABC = the given polygon. *Constr.*

The reqd. rect. can now be drawn by the method of the previous proposition.

PROPOSITION 24. PROBLEM.

To describe a parallelogram equal to a given triangle and having an angle equal to a given angle.



Let ABC be the given △, and D the given ∠.

It is required to construct a par^m equal to the △ABC, *so as to have an* ∠ *equal to* D.

Bisect BC at E. (I. 21.)

At E make the ∠CEH equal to the ∠D. (I. 25.)

Through C draw CK ‖ to EH, (I. 26.)

and through A draw AHK ‖ to BC meeting EH, CK at H, K respectively. CEHK *will be the required par^m.*

The △s ABE, AEC, on equal bases and between parallels, are equal; ∴ the △ABC is double of the △AEC.

But the par^m HECK is double of the △AEC; (II. 14.)

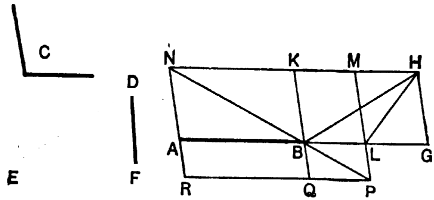
∴ the par^m HECK = the △ABC.

Moreover, it contains the ∠HEC equal to the ∠D;

∴ it has been constructed as required.

Proposition 25. Problem.

To describe a parallelogram which shall have a given straight line for one side, an angle equal to a given angle, and its area equal to a given triangle.



Let AB be the given str. line, C the given ∠, DEF the given △.

It is required to describe a par^m equal to △DEF, *having* AB *for one side, and having an* ∠ *equal to* ∠C.

On AB produced describe the △BGH having its sides respectively equal to those of △EFD. (I. 30.)

Make ∠GBK equal to ∠C. (I. 25.)

Bisect BG at L. (I. 21.)

Through H draw KMH ∥ to BG, (I. 26.)

and through L draw LM ∥ to BK,

and through A draw AN ∥ to BK or LM to meet KMH in N.

Join NB.

NB produced must meet ML produced, since AN and LM are parallel.

Let them meet at P.

Through P draw PQR ∥ to AB to meet KB and NA produced in Q and R.

ABQR shall be the figure required.

Join LH.

RB, BM, and RM are par^{ms} by constr.;

∴ the compt. RB = the compt. BM (II. 15.)

= twice the △BLH (II. 14.)

= △BHG (II. 11, *Cor.* 2.)

= △DEF. *Constr.*

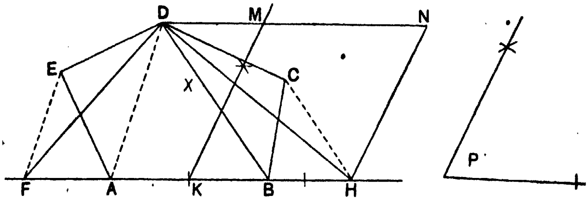
Also ∠ABQ = vertically opp. ∠LBK (I. 3.)

= ∠C. *Constr.*

∴ a par^m AQBR has been described, etc. Q.E.F.

PROPOSITION 26. PROBLEM.

To describe a parallelogram equal to a given rectilineal figure and having an angle equal to a given angle.



Let ABCDE be the given figure and P the given angle.
Reduce the figure ABCDE to the △FDH of equal area. (II. 21.)
Bisect FH at K, and make the ∠HKM equal to the given ∠P. (I. 25.)

Through D draw DMN parallel to FH,
and through H draw HN parallel to KM, (I. 26.)

The △s DFK, DKH on equal bases FK, KH and of the same altitude are equal in area.

∴ the △DFH = twice the △DKH
= the par^m HKMN. (II. 14.)
But the △DFH = the polygon ABCDE; *Constr.*
∴ the par^m HKMN = the polygon ABCDE. Q.E.F.

A **median** of a △ is a str. line drawn from an angular pt. to the mid. pt. of the opposite side.

PROPOSITION 27. THEOREM.

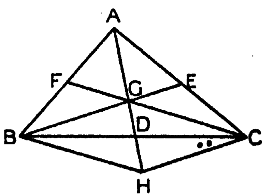
The medians of a triangle are concurrent, and their point of intersection is one of the points of trisection of each median.

Let the two medians BE, CF of the △ABC meet at G.

Produce AG to H making GH = GA.

If GH cuts BC at D, *it is reqd. to prove that* BD = DC.

Join BH, CH.



AF = FB *given.* and AG = GH; *Constr.*

∴ BH is ∥ to FG. (II. 6.)

Similarly CH is || to EG;

∴ BGCH is a parm.

But the diagonals of a parm bisect one another;

∴ BD = DC, which proves the first part of the proposition.

Also $GD = \frac{1}{2}GH = \frac{1}{2}AG$.

Similarly $GF = \frac{1}{2}GC$ and $GE = \frac{1}{2}BG$, which proves the second part of the proposition.

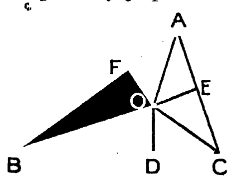
Centroid.—The point of intersection of the medians of a triangle is called the **Centroid** of the triangle.

Proposition 28. Theorem.

The bisectors of the angles of a triangle are concurrent.

Let two bisectors AO, BO of the ∠s A and B of the △ABC meet at O.

Draw OD, OE, OF respectively perpr to BC, CA and AB.



In the △s BOD, BOF, (1) OB is common,

(2) ∠DBO = ∠FBO. *Given.*

(3) ∠BDO = ∠BFO; *Cons.*

∴ OD = OF. (I. 10.)

In the same way, from the △s AOE, AOF,

OE = OF;

∴ OD = OE.

Then in the rt. angled △s ODC, OEC,

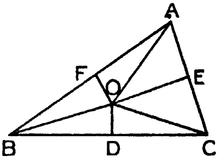
OC is common, and OD = OE;

∴ the △s are congruent, and ∠OCD = ∠OCE, (I. 14.)

which proves the proposition.

PROPOSITION 29. THEOREM.

The perpendiculars to the sides of a triangle at their middle points are concurrent.



In the $\triangle$ABC, let the perp^r bisectors OD and OE of BC and AC meet at O.

Join OA, OB, OC.

In the $\triangle$s BDO, CDO,

(1) BD = DC,

(2) OD is common,

(3) $\angle$BDO = $\angle$CDO ; (*rt.* $\angle$*s.*)

$\therefore$ OB = OC. (I. 9.)

Similarly OA = OC ; $\therefore$ OA = OB.

Draw OF perp^r to AB.

Then in the rt. angled $\triangle$s AFO, BFO,

The hypot. OA = hypot. OB,

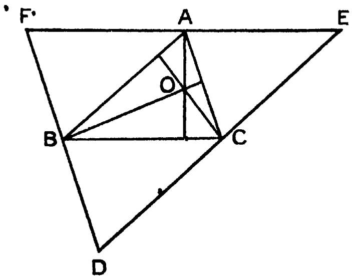
and OF is common ;

$\therefore$ AF = BF. (I. 14.)

which proves the proposition.

PROPOSITION 30. THEOREM.

The perpendiculars drawn from the angular points of a triangle to the opposite sides (sometimes called the **altitudes** *of the triangle) meet at a point, which is called the* **Orthocentre.**



Through the angular points A, B, C of the $\triangle$ABC draw EF, FD, DE respectively $\parallel$ to the opp. sides.

FACB is a par^{m}; $\therefore$ AF = BC. EABC is a par^{m}; $\therefore$ AE = BC.

$\therefore$ AF = AE.

Similarly DC = CE and DB = BF.

Hence the *altitudes* of the $\triangle$ABC are perp^{r} to the sides of the $\triangle$DEF at their mid. pts.

$\therefore$ these altitudes are concurrent. (II. 29.)

LOCI.

DEF. *If a point may move but is subject to certain restrictions, the path which it traces out is called its* **locus.**

For instance, if a point moves at a constant distance from a given point, its locus is a circle with the second given point for centre.

When the locus of a point is said to be a circle, the circumference of the circle is meant.

When a point moves at a constant perpendicular distance from a given straight line and always on the same side of it, its locus is a straight line parallel to the given straight line.

This can easily be verified on squared paper.

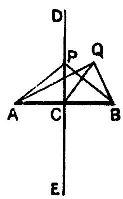
If the moving point may be on either side of the given line, the locus will be two straight lines, each parallel to the given line.

OX *is a given straight line, and* PN *is perpendicular to it. If* P *moves so that* PN *is always equal to* ON, *find, with the help of squared paper, the locus of* P.

OX *is a given straight line and* PN *is perpendicular to it. If* PN *is always equal to* 2ON, *find, on squared paper, the locus of* P.

PROPOSITION 31. THEOREM.

The locus of a point equidistant from two fixed points is a straight line bisecting at right angles the straight line joining the two fixed points.



Let A, B be the given points.

Suppose DCE drawn so as to bisect AB at rt. $\angle$s at C.

Let P be any point in DCE. Join PA, PB.

Then in the $\triangle$s ACP, BCP,

(1) AC = BC,

(2) PC is common,

(3) the $\angle$ACP = the $\angle$BCP;

$\therefore$ AP = BP. (I. 9.)

Similarly any point in DCE satisfies the given condition.

Also we have to prove that no point outside DCE satisfies the condition.

Let Q be any pt. outside DE. Join AQ, BQ, CQ.

In the $\triangle$s ACQ, BCQ, AC = BC,

CQ is common;

and $\angle$ ACQ is gr. than $\angle$ BCQ; *Obtuse $\angle$ gr. than acute $\angle$.*

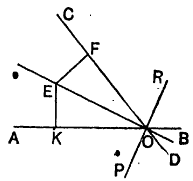
$\therefore$ AQ is gr. than BQ, (I. 18.)

i.e. Q is not a pt. on the locus.

$\therefore$ the locus is the str. line bisecting AB at rt. $\angle$s and produced infinitely both ways.

Proposition 32. Theorem.

The locus of a point which is equidistant from two given intersecting straight lines is the bisectors of the angles between the straight lines.



Let AOB, COD be two str. lines intersecting at O.

Take any pt. E on the bisector of the vertically opp. $\angle$s AOC, BOD.

Suppose EF drawn perp^r to CD, and EK perp^r to AB.

Then in the $\triangle$s OFE, OKE,

(1) the $\angle$ FOE = the $\angle$ KOE,

(2) the $\angle$ OFE = the $\angle$ OKE (rt. $\angle$s),

(3) OE is common to both $\triangle$s;

$\therefore$ EF = EK. (I. 10.)

In the same way, if H is any pt. on the bisector POR of the $\angle$s AOD, BOC, it may be shown that H is equidistant from the given lines.

Conversely, let E be any pt. within the $\angle$ AOC such that the perpendiculars EK, EF to OA and OB respectively are equal.

In the rt. $\angle$d $\triangle$s EKO, EFO,

the hypotenuse OE is common,

EK = EF;

$\therefore$ $\angle$ EOK = $\angle$ EOF, (I. 14.)

i.e. the pt. E lies on the bisector of the $\angle$ AOC.

In the same way if a pt. H within the $\angle$ AOD or $\angle$ COB satisfies the reqd. condition, we can prove that H lies on the bisector of these angles.

We have now proved that:

(1) If E lies on either bisector of the angles, it is equidistant from the given lines.

(2) If E is equidistant from the given lines, it must lie on one of the bisectors.

∴ the locus of pt. which is equidistant, etc. Q.E.D.

EXERCISES.

On Loci.

1. Find the locus of the centre of a circle of given radius which rolls on the outside of a given circle. XIV. 1.

2. Find the locus of the centre of a circle of given radius which rolls on the inside of a given circle. XIV. 2.

3. Find experimentally, on 'squared' paper, the locus of a point which moves so that its distance from one straight line is double its distance from a perpendicular straight line. XIV. 3.

4. Find experimentally, without proof, the locus of the middle point of a straight line drawn from a given point to meet a given straight line of unlimited length. XIV. 4.

5. What is the locus of the middle points of the radii of a given circle? XIV. 5.

6. A and B are points on two parallel straight lines; prove that the middle point of AB is equidistant from the given straight lines. What do you infer about its locus? XIV. 6.

IMPORTANT EXERCISES.

1. Bisect a given triangle by a straight line drawn from a given point in one side. XXIII. 1.

(If ABC be the given △, and D the given point in AB, bisect AC at E and draw BF parallel to DE to meet AC in F. The △ADF will be the △ reqd.

This construction fails if AD is less than DB. Solve the problem in such a case.)

2. Through any point D in a side BC (or BC produced) of a triangle ABC, draw a straight line to meet AC (produced if necessary) in E, such that the triangle CDE is equal in area to the triangle ABC. XXIII. 2.

(Draw BE parallel to AD.)

3. If AD, CF, BE be parallels meeting two straight lines AB, DE, and if C be the middle point of AB, then CF is half the sum of AD and BE, unless the point of intersection of AB and DE lies between A and B, in which case CF is half the difference of AD and BE. XXIII. 3.

4. Divide a given straight line into two parts so that the square on one part may be twice the square on the other. XXIII. 9.

(Use the fact that the square on the hypotenuse of an isosceles rt.-angled Δ = twice the sq. on either side.)

5. The bisectors of two exterior angles and the bisector of the third angle of a triangle are concurrent. XXIII. 10.

MENSURATION EXAMPLES.

1. Draw a straight line $\sqrt{8}$ inches long. Measure its length as exactly as you can. XXV. 6.

2. Draw a straight line of length $\sqrt{5}$ centimetres, and measure it. XXV. 7.

3. Taking any straight line as unit length, obtain a straight line whose length is $\sqrt{7}$. Explain your construction, and measure the line. XXV. 8.

4. A ladder, 50 ft. long, rests against the top of a vertical wall BC, 32 ft. high, with one extremity A on horizontal ground. If AB, the distance of the foot of the ladder from the bottom of the wall, be 24 ft., find how far the ladder projects beyond the top of the wall. XXV. 9.

5. Make a straight line 5 cm. long, and find by construction another line the square on which is twice the square on the first line. Measure the line and write down its length. XXV. 10.

6. A string, 108 ft. long, has its extremities fixed at points 72 ft. apart. If it be held at a point 30 ft. from one end and drawn aside until tight, show that in this position the shorter portion of the string will be at right angles to the line joining the fixed points. XXV. 11.

7. Draw a right-angled triangle on a hypotenuse 6 cm. long, having one angle equal to 60 degrees. Prove, geometrically, that its shortest side is 3 cm. long, and measure, as exactly as you can, its other side. XXV. 14.

8. A quadrilateral has diagonals 30 and 40 ft. long respectively, and at right angles to one another: find the area of the figure. XXV. 15.

9. Find the area of a triangle having two sides 4 in. and 6 in. long, and the included angle 30 degrees. XXV. 16.

10. The area of a square is 15 acres: calculate, correct to a link, the length of its side. XXV. 17.

11. Find in square feet the area of a square whose diagonal is 6 feet long. XXV. 18.

12. The diagonal of a rectangle is 8 feet long, and one of its sides is 5 feet long. Find, to the nearest square foot, its area. XXV. 19.

13. How many tiles 3 inches square will it take to pave a hall 8 feet by 12 feet? XXV. 20.

14. It takes 1920 tiles 4 inches square to pave a hall 8 feet wide. Find its length. XXV. 21.

15. In a right-angled triangle the area is one quarter of an acre, and one side is 22 yards long. Find the length of the other side. XXV. 22.

16. Find the area of a right-angled triangle whose hypotenuse is 39 feet and one side 15 feet long. XXV. 23.

17. In the triangle ABC, the base BC = 12 feet, AC = 5 feet, and the $\angle$ ACB = 45 degrees. Find its approximate altitude and area. XXV. 24.

18. One side of a triangle is 12 feet, and the perpendicular on it from the opposite angular point is 6 feet long. Find the length of the perpendicular from the opposite angular point on another side which is 14 feet long. XXV. 25.

19. Draw a triangle whose sides are 5 cm., 5·6 cm., and 6·4 cm. long. Draw and measure an altitude of the triangle, and hence calculate its approximate area. XXV. 26.

20. Calculate to the nearest square foot the area of a rhombus whose sides are 6 feet long, and one of whose angles is 45 degrees. XXV. 27.

21. Find to two decimal places the diagonal of a square whose side is 18 feet. XXV. 28.

22. A boat making 10 miles an hour steams for 3 hours due east, and then for 4 hours due south. How far is it then from its starting point? XXV. 29.

23. Find to the nearest link the diagonal of a square whose side is 4 chains 10 links. XXV. 30.

24. A ladder 31 feet long is placed so that it reaches a point in the front of a house 23 feet above the ground. How far is its foot from the front of the house? Calculate your result correctly to two decimal places in feet. XXV. 31.

25. The length of the shadow of a tower is 200 feet when the altitude of the sun is 30 degrees. Calculate correctly to two places of decimals the height of the tower. XXV. 32.

26. A boy, 4 feet high, stands at a distance of 10 feet from a lamp-post, and his shadow is observed to be 10 feet long. Draw a diagram to scale (5 feet to an inch), and find the height of the lamp-post. XXV. 33.

27. Find the length of the hypotenuse of a right-angled triangle which has the other two sides $m^2 - n^2$ and $2mn$ cm. long. Show that if $m = n + 1$ the hypotenuse differs from another side by 1 centimetre, and find the three sides when $m = 13$, $n = 12$. XXV. 34.

28. The parallel sides of a trapezium are 9 and 30 feet long, and the other sides are 17 and 10 feet long. Find its area. XXV. 35.

29. There is a piece of ground in the form of a trapezium, the lengths of the parallel sides of which are 20 and 34 yards, and the lengths of the other two sides 15 and 13 yards. Find its area. XXV. 38.

30. A straight road rises at an angle 30°. How far up it must you go to be 20 ft. above the starting-point? What horizontal distance will you be from the start? XXV. 57.

31. What area does a roller of width 3 ft. 6 in. and diameter 2 ft. 6 in. pass over in 100 revolutions? XXV. 60.

32. The area of a right-angled triangle is 98 sq. cm., and one of the sides containing the right angle is twice the other. Find the length of the hypotenuse and the perpendicular from the right angle to it. XXV. 61.

33. A man wishes to find the breadth AB of a river. He walks 300 feet along AC at right angles to AB, and finds the angle ACB to be 65°. He then calculates the required breadth to be 640 feet. Estimate, on squared paper, his error. XXV. 39.

34. At a point 200 yards from the base of a tower, the angle of elevation of its top is found to be 25°. Find, by means of squared paper, the height of the tower. How could you ascertain the height of a tree by means of an isosceles right-angled triangle? XXV. 40.

35. A land surveyor wishing to find the width of a pond, places two posts at A and B, two points on opposite banks of it, and from a point C makes the following measurements:

$$AC = 25 \text{ yards}, \quad BC = 20 \text{ yards}, \quad \text{and the } \angle ACB = 40°.$$

Find the width of the pond. XXV. 41.

36. Construct a rhombus whose diagonals are 64 and 48 millimetres long. XXV. 42.

37. Construct a parallelogram having one side 65 mm. long, one angle equal to 70°, and one diagonal 85 mm. long. XXV. 43.

38. On a base 12 cm. long construct a trapezium having an altitude of 3 cm., and each of its equal sides 6 cm. long. XXV. 44.

39. The diagonals of a parallelogram are 60 and 76 mm. long, and one side is 33 mm. long: construct it. XXV. 45.

40. Draw two lines at an angle of 65°, and find a point 1 inch distant from one and 2 inches from the other. XXV. 46.

41. Draw a plan, to scale, of the floor of a room, 16 feet wide, and 24 feet long. Find the distance between two opposite corners. XXV. 47.

42. On one side of a square describe an isosceles triangle equal in area to the square. XXV. 49.

43. A rhombus has an angle of 45°, and its sides are rods of 4 inches hinged together at their ends. How much must one diagonal be shortened to make it a square? Find also the increase of area. XXV. 51.

44. Two sides of a triangle are 4 and $4\sqrt{3}$, and the included angle is 30°. Find graphically the hypotenuse of an equal isosceles right-angled triangle. XXV. 59.

45. A gardener is told to enlarge a square flower bed of 12 feet side into one twice as great in area. He makes the side 17 feet. How much shorter ought the side to be? XXV. 52.

46. A pole 60 feet high stands 40 feet from a house. At what point may it break so as just to miss a window, the bottom of which is 40 feet from the ground? XXV. 53.

47. Describe a regular hexagon on a base 2 inches. Reduce it to an equal triangle of the same altitude, and measure the base. XXV. 54.

48. The parallel sides of a trapezium are 5 and 7 cm., and the other sides are equal, and each 3 cm. Find the distance between the parallel sides. XXV. 55.

49. A straight line is 12 inches long. Divide it into 4 lengths which shall make a rectangle $6\frac{3}{4}$ sq. inches in area. XXV. 56.

50. A rectangle containing 640 sq. cm. has one side 24 cm. longer than another. Find the sides. XXV. 58.

51. A quadrilateral has a diagonal of 84 feet, and the distances of the vertices from this diagonal are 37 and 41 feet. Find its area. XXV. 65.

52. A trapezium has its parallel sides 3 and 28 cm., and the other sides 25, 30 cm. Find its area. XXV. 66.

53. A ladder reaches a window 24 feet high on one side of a street. When turned over through a right angle it reaches one 18 feet high on the other side of the street. Find the width of the street and the length of the ladder. XXV. 67.

54. The sides of a triangle are 17, 21, 28. Find, by measurement, the length of the median drawn to the greatest side. XXV. 68.

55. One side of a regular hexagon is 16. Find the area to one place of decimals. XXV. 69.

56. Draw plans of the under-mentioned pieces of ground, and find their areas: XXV. 70.

(a) YARDS

	to G	
	204	
to F 94	198	
	122	10 to E
to D 64	117	
to C 14	88	
	63	70 to B
	From A	

(b) YARDS

	to G	
	600	80 to H
to F 140	560	
to E 150	480	
	470	200 to D
to C 100	380	
	100	150 to B
	From A	

57. In a four-sided field ABCD, AB and AD are each 96 yards; BC = 64 yards; CD = 48 yards and the $\angle$BCD is a right angle. Find its area to the nearest sq. yd. XXV. 73.

58. A gunner at P wishes to know the distance of a point E held by the enemy. Suppose that R is a third point, and that he knows

PR = 4325 yards, EPR = 70°, ERP = 64°,

and find the distance PE by measurement. XXV. 74.

59. A duck enters the water at an angle of 30° to the horizontal and continues that course below the surface for 18 feet. Another straight course of 18 feet brings it to the surface. Calculate how far off it emerges, and what is the greatest depth reached. XXV. 76.

60. From a rectangular field part is to be taken for a building and another part for a road. The part for the building is a triangle in one corner of the field, measuring 150 metres along each of two sides of the field. The road is to be 11·5 metres wide and to have the hypotenuse of the triangle for one side. Find the cost of the ground taken at 10s. a sq. metre. XXV. 77.

61. A rectilineal figure stands on a base of 5 inches. Its heights measured at intervals of 1 inch are 0, 3, 2, 4, $4\frac{1}{2}$, 3 inches. Find its area. XXV. 80.

62. A tower is 48 feet high. Its section is a square of area 25 sq. feet. Find the length of the shortest rope which will reach from top to bottom, taking one turn round the tower. XXV. 81.

MISCELLANEOUS EXERCISES.

1. The sides of a triangle are a, b, c inches. State:

(1) Which of the following relations are true for all triangles,

(2) Which are untrue for all triangles,

(3) Which are true for some and untrue for others:

$$a+b>c, \quad a+b=c, \quad a+b<c,$$
$$a^2+b^2=c^2.$$

XXVI. 1.

2. ABCD is a quadrilateral figure. The angle ABC is a right angle, the diagonals AC, BD are at right angles, and

$$AC = BD = DC.$$

Construct the figure accurately, and prove that 2AB = BC. XXVI. 2.

3. Construct a square which shall have an extremity of one of its diagonals at a given point, and the extremities of the other diagonal on a given straight line. XXVI. 6.

4. D is any point in the side BC of an equilateral triangle ABC; if AD be joined and an equilateral triangle ADE be described upon it, then will either BD be equal to CE, or BE be equal to CD. XXVI. 7.

5. ABCD is a quadrilateral having BC parallel to AD, and E is the mid. point of DC. Show that the triangle AEB is half the quadrilateral. XXVI. 25.

6. Through P the middle point of the base BC of a triangle ABC, a straight line DPE is drawn cutting one of the sides in D and the other side produced in E; prove that the triangle ADE is greater than the triangle ABC. (The point E is below the base.) XXVI. 10.

7 If lengths CD, CE, each equal to a line CA, be measured off on any straight line through C, and AD, AE be joined, prove that DAE is a rt. angle. XXVI. 16.

8. If a quadrilateral be bisected by each of its diagonals it is a parallelogram. XXVI. 24.

9. Describe a rhombus equal to a rectangle 4 cm. by 5 cm.

(*Note.*—The diagonals of a rhombus bisect each other at right angles.)

10. If from any point within a rectangle lines be drawn to the angular points, the sums of the squares of those which are drawn to the opposite angles are equal. XXVI. 28.

11. Construct a rectangle equal to 16 sq. cm. if the length of one side of the rectangle is to be 2·5 cm. XXVI. 30.

12. The straight lines joining the ends of one diagonal of a parm to the mid. pts. of either pair of opposite sides trisect the other diagonal. XXVI. 31.

13. The str. lines joining any point within a triangle to the vertices are together less than the perimeter, but greater than the semi-perimeter of the triangle. XXVI. 32.

14. Construct a triangle whose angles shall be proportional to 1, 2, 3. XXVI. 17.

15. If a straight line EF drawn || to the diagonal AC of a parm ABCD meets AD, DC, or those produced, in E, F respectively, the $\triangle$ABE = the $\triangle$BCF. XXVI. 18.

16. The difference between the squares on the segments of the base made by the perpendicular from the vertex of a triangle = the difference between the squares on the lines joining the ends of the base to any point on the perpendicular. XXVI. 15.

Hence prove the altitudes of a triangle are concurrent.

17. Construct a square that shall be one-third of a given square. XXVI. 34.

18. Construct a parallelogram which shall be double a given quadrilateral.

19. Construct a square equal to a rectangle 7 cm. by 4 cm.

20. Bisect a given quadrilateral by a straight line from one vertex. (Reduce to an equal Δ.)

21. On the same side of the str. line ABC equal rectangles ABDE, ACFH are described. Prove that BH is $\parallel$ to FD. XXVI. 48.

22. ABCD is a quadrilateral of any shape. Apply to BC a parallelogram of equal area having the angle ABC for its angle at B (*i.e.* having one of its sides along BA). XXVI. 54.

23. A piece of paper is in the form of two squares, of which two sides are in the same straight line. Show how to divide the paper by two straight cuts into three pieces which will form a square. XXVI. 26.

24 Convert a square into a regular octagon. [Draw circles with the half diagonals as radii.] Find the length of the sides of the octagon so formed.

BOOK III

DEFINITIONS

1. A **circle** is a *plane figure* bounded by one line called the circumference, and is such that all straight lines drawn from a certain point within the figure to the circumference are equal.

Note.—(*a*) According to the definition a circle is the **figure** bounded by the circumference, but the circumference itself is sometimes called the circle; *e.g.* in such a statement as "The locus of a point at a given distance from a given point is a circle."

(*b*) *If two circles have equal radii they are equal.*

This may be proved at once by placing one circle on the other with centre on centre; for, in consequence of the equality of the radii, the circumferences must entirely coincide.

Thus also two concentric circles either coincide entirely or do not meet at all.

2. A **chord** of a circle is the finite straight line joining any two points on the circumference.

3. Any straight line cutting the circumference of a circle at two points is said to be a **secant** of the circle.

4. An **arc** of a circle is part of the circumference.

5. A **segment** of a circle is the *figure* contained by a chord and the arc which it cuts off.

6. A **sector** of a circle is the *figure* contained by two radii and the arc between them.

7. The **angle in a segment** is the angle contained by two straight lines drawn from any one point of the bounding arc to the ends of the chord of the segment.

Similar segments are those which contain equal angles.

8. When each vertex of a rectilineal figure lies on the circumference of a circle, the figure is said to be inscribed in the circle, and the circle is said to be described about the figure.

(The circle is called the **circumscribed** circle or **circum-circle**.)

When each side of a rectilineal figure touches a circle, the figure is said to be described about the circle, and the circle is said to be inscribed in the figure. (The **in-circle.**)

9. **Escribed circle.**—If a circle touches one side of a triangle and the other two sides produced, it is said to be an **escribed circle** of the triangle.

10. A figure is said to be *symmetrical about a line* if the part on the one side coincides with the part on the other side when the figure is folded about that line.

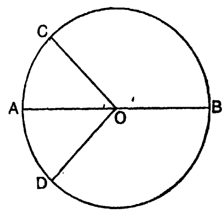
The crease is called an *axis of symmetry*.

E.g. An isosceles △ is symmetrical about the bisector of its vertical angle.

In any symmetrical figure, two points are called *images* (one of the other), or *symmetrically opposite points*, when the symmetrical folding brings them together. They are evidently equidistant from the crease.

Before proceeding further, the student should work some examples on " Symmetrical Figures " and " Lines of Symmetry " (see p. 15).

A circle is symmetrical about any diameter.



Describe any circle, centre O, and draw any diameter, AOB. Take any point C on the circumference, and join OC.

Fold the circle about the straight line AB, and let the point C, in its new position, fall at D. Join OD.

Then OC coincides with OD ;

∴ OD = OC = the radius of the circle;

∴ the point D lies on the circumference of the circle.

Similarly any other point in the arc ACB will, when the circle is folded, fall on the arc ADB.

∴ the two semicircles coincide, and the circle is therefore symmetrical about the diameter.

Thus we see that a diameter bisects a circle.

Proposition 1. Theorem.

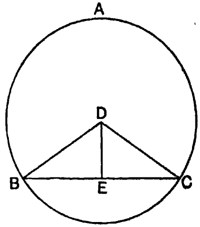
The line drawn from the centre of a circle to bisect a chord is perpendicular to it; and the perpendicular drawn to a chord from the centre bisects the chord.

Let BC be any chord of a circle ABC, and D its centre.

(i) Let DE bisect BC at E.

It is reqd. to prove that DE *is perp*[r] *to* BC.

Join DB, DC.



In the $\triangle$s DEB, DEC,

(1) DB = DC, *Radii.*

(2) DE is common,

(3) BE = EC;

$\therefore$ the $\angle$DEB = the $\angle$DEC. (I. 13.)

But these are adj. $\angle$s, and therefore rt. $\angle$s. Q.E.D.

(ii) Let DE be perp[r] to BC;

then DE *shall bisect* BC.

In the $\triangle$s DEB, DEC,

(1) the $\angle$DBE = the $\angle$DCE ($\angle$s at the base of an isosceles $\triangle$),

(2) the $\angle$DEB = the $\angle$DEC (right angles),

(3) DE is common;

$\therefore$ BE = EC. (I. 10.)

Q.E.D.

Corollary. From this it follows that the straight line which bisects at right angles a chord of a circle passes through the centre of the circle.

EXERCISES.

1. In a circle of radius 5 cm. there is a chord of length 6 cm. How far is it from the centre?

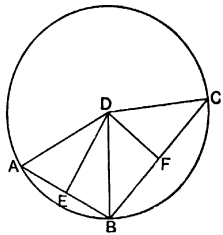
2. From a point outside a circle of radius 13 cm. a straight line is drawn at a perpendicular distance of 12 cm. from the centre. Find the length of the intercepted chord.

PROPOSITION 2. THEOREM.

There is one circle, and only one circle, which passes through three given points not in the same straight line.

Let A, B, C be three pts. not in the same str. line.

It is reqd. to prove that one circle, and only one, can be drawn passing through the pts. A, B, C.



Through E and F the mid. pts. of AB and BC, draw perp^rs. These will meet; for A, B, and C are not in a str. line.

Let them meet at D. Join DA, DB, DC.

ED bisects AB at rt. $\angle$s; $\therefore$ DA = DB. (II. 31.)

FD „ BC „ „ $\therefore$ DB = DC. (II. 31.)

$\therefore$ DA = DB = DC,

and a circle described with centre D and rad. DA will pass through B and C.

Also D is the only pt. on both the perp^r bisectors of AB and BC;

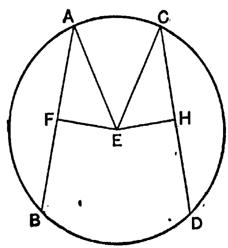
$\therefore$ only one circle can be drawn through A, B and C. Q.E.D.

N.B.—We have proved above that:

There is one, and only one, point equidistant from three given non-collinear points.

PROPOSITION 3. THEOREM.

Chords equally distant from the centre of a circle are equal, and conversely. Also of two chords, that which is nearer the centre is greater than that which is more remote, and conversely.



Let AB, CD be two chords of a circle; EF, EH the perp^rs from the centre upon them.

Join AE, EC.

Then AF = $\frac{1}{2}$AB, and CH = $\frac{1}{2}$CD. (III. 1.)

Also, since the angles at F and H are rt. $\angle$s,

$$AF^2 + EF^2 = AE^2 = CE^2 = CH^2 + EH^2 \ldots\ldots (1) \quad \text{(II. 16.)}$$

(*a*) If EF = EH, AF = CH ;

$\therefore$ AB = CD.

Conversely, if AB = CD, AF = CH (halves of equals) ;

$\therefore$ EF = EH, from (1).

(*b*) If EF is less than EH, AF is gr. than CH ;

$\therefore$ AB is gr. than CD.

Conversely, if AB is gr. than CD, AF is gr. than CH ;

$\therefore$ EF is less than EH, from (1).

This is true not only for one circle, but also for equal circles.

EXERCISES.

1. Equal chords of a circle subtend equal angles at the centre. XXVII. 1.

2. The diameter of a circle is 30 inches, and a chord is 24 inches long. Find the distance of the chord from the centre. XXVII. 11.

3. Two chords of a circle cannot bisect each other unless they are diameters. XXVII. 4.

4. Two circles of radii 15 and 20 cm. have a common chord 24 cm. long. Find the distance between their centres. XXVII. 5.

5. The locus of the mid. points of parallel chords is a diameter. XXVII. 7.

6. If the line joining the mid. points of two chords is perpʳ to one, it is perpʳ to the other. XXVII. 9.

7. A straight line cuts two concentric circles. Prove that the parts between the circles are equal XXVII. 10.

8. If two circles intersect in A, a line drawn through A terminated by the circumferences and parallel to the line of centres is double of the line joining the centres. XXVII. 13.

9. Two circles intersect at A, B, and are met in C, D, E, F by a parallel to AB. Prove that CD is equal to EF. XXVII. 14.

10. AB, AC are the equal sides of an isosceles triangle; a circle whose centre is A cuts the base (or base produced) in D, E; prove that BD = EC. XXVII. 17.

11. If two equal circles be drawn, each passing through the centre of the other, the square on the common chord is three times the square on either radius. XXVII. 18.

12. If two circles intersect, any two parallel lines drawn through their common points and terminated by the circles are equal. XXVII. 19.

Proposition 4. Theorem.

The angle at the centre of a circle is double of the angle at the circumference standing on the same arc.

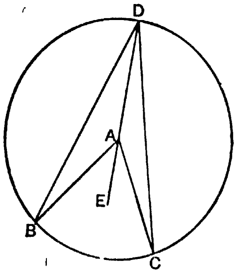


Fig. 1.

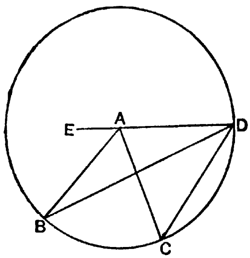


Fig. 2.

Let BAC be an angle at the centre of the circle BDC, and BDC an angle at the circumference standing on the same arc BC.

It is reqd. to prove that the $\angle$BAC *is double of the* $\angle$BDC.

In Fig. 1 A falls within the $\angle$BDC, in Fig. 2 outside it.

Produce DA to E.

AB = AD (radii);

$\therefore$ the $\angle$ADB = the $\angle$ABD. (I. 11.)

But the ext. $\angle$EAB

= the $\angle$ABD + the $\angle$ADB; (I. 7.)

$\therefore$ the $\angle$EAB is double of the $\angle$ADB.

Similarly the $\angle$EAC is double of the $\angle$ADC.

By adding these results in Fig. 1 and subtracting them in Fig. 2 it follows that the $\angle$BAC is double of the $\angle$BDC in either case.

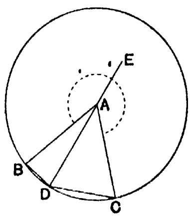


Fig. 3.

The proof for Fig. 1 will hold if the angle BAC is greater than two right angles. See Fig. 3. Q.E.D.

Definition. An angle greater than two right angles is called a **reflex** angle.

Proposition 5. Theorem.

Angles in the same segment of a circle are equal.

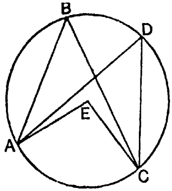


Fig. 1.

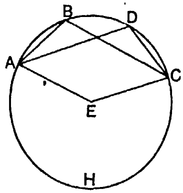


Fig. 2.

Let ABC, ADC be any two angles in the same segment ABDC.

It is reqd. to prove that the ∠ABC = *the* ∠ADC.

Let E be the centre. Join AE, EC.

In Fig. 1,

the ∠AEC is double of the ∠ABC on the same arc AC, (III. 4.)

the ∠AEC ,, ∠ADC ,, ,, (III. 4.)

∴ the ∠ABC = the ∠ADC.

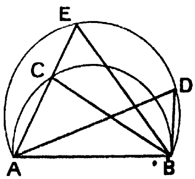
In Fig. 2 the reflex angle AEC is double of the ∠ABC on the same arc AHC, and is also double of the ∠ADC. (III. 4.)

∴ the ∠ABC = the ∠ADC. Q.E.D.

Concyclic Points.—Points which lie on the circumference of the same circle are said to be **concyclic.**

Proposition 6. Theorem.

If the straight line joining two points subtends equal angles at two other points on the same side of it, the four points are concyclic.



Let the str. line AB subtend equal angles ACB, ADB at the points C, D on the same side of AB.

It is reqd. to prove that A, C, D, B *are concyclic.*

Suppose the circles through A, C, B and A, D, B to be drawn. If the arcs ACB, ADB do not coincide, one of them must be wholly within the other, since two circles cannot cut at more than two points.

Suppose ACB to be the inner arc, and produce AC to meet the outer arc at E.

Join BE.

Then in the $\triangle$BEC, the ext. $\angle$ACB is greater than the int. opp. $\angle$CEB.

But $\angle$ACB $=$ $\angle$ADB *Given.*

$=$ $\angle$AEB in the same segment,

which is absurd.

$\therefore$ the arcs ACB, ADB must coincide, *i.e.* the points A, C, D, B are concyclic. Q.E.D.

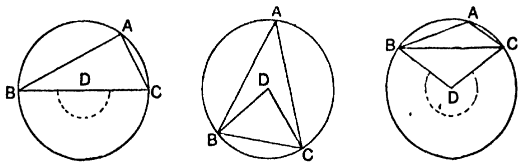
(The above proof is that recommended in the *Report of the I.A.A.M. on the Teaching of Elementary Geometry.*)

Cor. From this proposition we see that:

If a triangle be described on one side of a given base, and have a given vertical angle, the locus of its vertex is an arc of a circle on the given base.

Proposition 7. Theorem.

The angle in a semi-circle is a right angle; the angle in a segment greater than a semi-circle is less than a right angle; and the angle in a segment less than a semi-circle is greater than a right angle.



Let D be the centre of the circle.

(1) When BAC is a semi-circle.

Then the $\angle$BAC $=$ half the straight $\angle$BDC (III. 4.)

$=\frac{1}{2}$ of two rt. $\angle$s

$=$ a rt. $\angle$.

(2) When BAC is a segment gr. than a semi-circle,

the $\angle$BDC is less than two rt. $\angle$s,

and the $\angle$BAC $=$ half the $\angle$BDC;

$\therefore$ the $\angle$BAC is less than a rt. $\angle$.

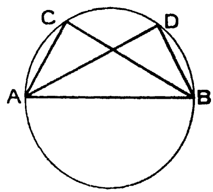
(3) When BAC is a segment less than a semi-circle, the reflex ∠BDC is gr. than two rt. ∠s,

and the ∠BAC = half the reflex ∠BDC;

∴ the ∠BAC is gr. than a rt. ∠.

PROPOSITION 8. THEOREM.

The circle described on the hypotenuse of a right-angled triangle as diameter passes through the opposite vertex.



Take D, any pt. on the circle described on the hypotenuse AB of the rt. angled △ABC as diameter, C and D lying on the same side of AB.

Join DA, DB.

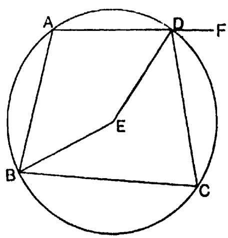
∠ADB in a semi-circle is a rt. ∠; (III. 7.)

∴ ∠ADB = ∠ACB;

∴ the four pts. A, C, D, B are concyclic, (III. 6.)
which proves the proposition.

PROPOSITION 9. THEOREM.

The opposite angles of a quadrilateral inscribed in a circle are together equal to two right angles.



[Such a quadrilateral is called a **cyclic** quadrilateral.]

Let ABC be a circle, and ABCD a quad[l] inscribed in it.

It is reqd. to prove that its opposite ∠s are supplementary.

Let E be the centre of the circle.

Join BE, ED.

The ∠BED = twice the ∠BCD on the same arc BAD. (III. 4.)

The reflex ∠BED = twice the ∠BAD on the same arc BCD. (III. 4.)

∴ the ∠BCD + the ∠BAD = half the ∠BED + half the reflex ∠BED
= half of 4 rt. angles
= 2 rt. angles.

Similarly the $\angle$ ABC + the $\angle$ ADC = 2 rt. angles;

$\therefore$ the opposite angles, etc. Q.E.D.

COROLLARY. *If one side of a cyclic quadrilateral is produced, the exterior angle so formed is equal to the interior opposite angle of the quadrilateral.*

In the above fig. produce AD to F.

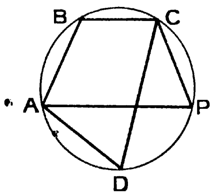
$\angle$ FDC = the supplement of $\angle$ ADC

= $\angle$ ABC. *Proved above.*

Q.E.D.

PROPOSITION 10. THEOREM.

If a pair of opposite angles of a quadrilateral are supplementary, its vertices are concyclic.



Let ABCD be a quad[l] in which the angles B and D are supplementary.

It is reqd. to prove that A, B, C, D *are concyclic.*

Suppose the circle through the pts. A, B, C to be drawn.

Take P any pt. on its circumference on the same side of the chord AC as D. Join PA, PC.

ABCP is a cyclic quad ;

$\therefore$ $\angle$s P and B are supplementary.

But $\angle$s D and B are supplementary; *Given.*

$\therefore$ $\angle$P = $\angle$D;

$\therefore$ A, D, P, C are concyclic; (III. 6.)

$\therefore$ the pt. D lies on the arc APC,

i.e. on the circle through A, B and C;

$\therefore$ A, B, C, D are concyclic. Q.E.D.

(The above proof is that suggested in the *Report of the I.A.A.M. on the Teaching of Elementary Geometry.*)

EXERCISES.

1. AB, CD are two chords of a circle intersecting in E; prove that the triangle AEC is equiangular to the triangle DEB. XXX. 1.

2. A triangle is inscribed in a circle (*i.e.* its vertices lie on the circumference); prove that the angles in the three segments exterior to the triangle are together equal to 4 right angles. XXX. 2.

3. A regular pentagon is inscribed in a circle; prove that the angle subtended at a point on the circumference by the nearest side is four times the angle subtended by each of the other sides. XXX. 3.

4. If a side of a quadrilateral be produced, and the exterior angle = the interior opposite angle, the quad[l] is cyclic. XXX. 7.

5. D is any point on the minor arc BC of a circle whose centre is A; CD is produced to E; prove that the angle BDE is half the angle BAC. XXX. 23.

6. If the sides AB, DC of a cyclic quadrilateral be produced to meet in E, the triangles EBC, EDA are equiangular to each other. XXX. 8.

7. In a triangle ABC, AD and BE are drawn perpendicular to the opposite sides: prove that the angle DEC is equal to the angle ABC. XXX. 16.

8. AD, BE are drawn perpendicular to sides BC, AC of the triangle ABC: prove that the angles ADE, ABE are equal. XXX. 17.

9. Two circles ABEC, ABFD intersect at A and B, and CAD and EBF are straight lines: prove that EC is parallel to DF. XXX. 18.

10. ACB, APB are two equal circles, the centre of APB being on the circumference of ACB; if any chord CA of ACB be produced to cut APB in P, the triangle PBC is equilateral. XXX. 19.

11. The straight lines which bisect any angle of a quadrilateral inscribed in a circle and the opposite exterior angle meet on the circumference. XXX. 22.

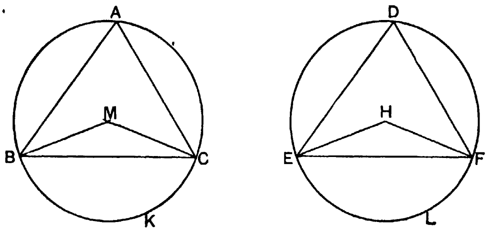
12. A circle of constant radius is drawn through O cutting two fixed straight lines OA, OB in P, R. Show that the chord PR is of constant length. XXX. 21.

13. Four circular coins, of different sizes, are placed upon a table so that each one touches two, and only two, of the remaining three; show that the four points of contact lie on a circle. XXX. 11.

14. A circle drawn through the mid. points of the sides of a triangle passes through the feet of the perpendiculars from the vertices to the sides. [If D, E, F are the mid. points opposite to A, B, C and AH the altitude, prove that EF subtends at both D and H an angle equal to A.] XXX. 4.

Proposition 11. Theorem.

In equal circles, arcs are equal if they are subtended by equal angles at the centres, or by equal angles at the circumferences.



Let ABKC, DELF be equal circles, M, H their centres; and let BAC, EDF be equal $\angle$s at the circumferences, and consequently BMC, EHF equal $\angle$s at the centres. (III. 4.)

It is reqd. to prove that the arc BKC = *the arc* ELF.

Apply the circle ABC to the circle DEF, placing M on H and MB along HE.

Now the $\angle$BMC = the $\angle$EHF; *Given.*

$\therefore$ MC falls on HF.

Also, since the circles have equal radii their circumferences coincide entirely.

$\therefore$ the arc BKC coincides with the arc ELF, and they are consequently equal. Q.E.D.

Corollary. In equal circles, if angles at the centres are equal the sectors thus cut off are equal.

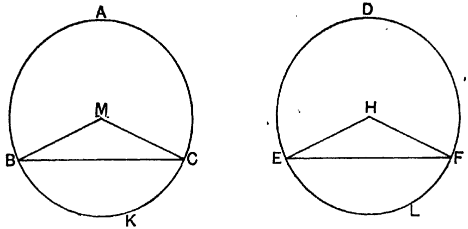
Note.—*Propositions* 11, 12, 13 *are also true if there be one circle instead of equal circles.*

Proposition 12. Theorem.

In equal circles, if there are equal arcs, the angles subtended by them at the centres are equal, and the chords of the arcs are also equal.

Let ABKC, DELF be equal circles with centres M, H; and let the arc BKC = the arc ELF.

It is reqd. to prove that the $\angle$BMC $=$ $\angle$EHF, *and the chord* BC $=$ *the chord* EF.



Apply the circle ABC to the circle DEF, placing M on H and MB along HE.

Then the circumferences must coincide throughout since the circles have equal radii.

But the arc BKC $=$ arc ELF ; *Given.*

$\therefore$ the point C coincides with the point F ;

$\therefore$ the $\angle$BMC coincides with the $\angle$EHF, and is therefore equal to it ; also the chord BC coincides with the chord EF, and is therefore equal to it. $\therefore$ in equal circles, etc. Q.E.D.

Cor. 1. In equal circles, the sectors on equal arcs are also equal.

Cor. 2. In equal circles, equal arcs subtend equal angles at the circumferences.

Proposition 13. Theorem.

In equal circles the arcs which are cut off by equal chords are equal, the major arc equal to major, and minor arc equal to minor.

Let ABKC, DELF be equal circles, and BC, EF equal chords.

It is reqd. to prove that the minor arc BKC $=$ *the minor arc* ELF, *and the major arc* BAC $=$ *the major arc* EDF.

Join BM, MC, EH, HF.

In the Δs BMC, EHF, (1) BM $=$ EH (radii of equal circles),

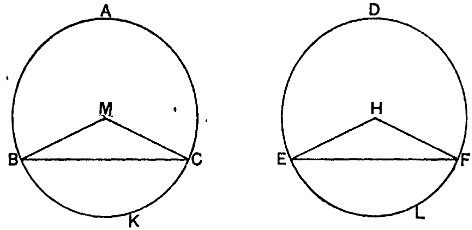
(2) MC $=$ HF (radii of equal circles),

(3) BC $=$ EF ; *Given.*

$\therefore$ the $\angle$BMC $=$ the $\angle$EHF. (I. 13.)

Apply the circle ABC to the circle DEF, placing M on H, and MB along HE.

Then the circumferences must coincide throughout, since the circles have equal radii.



But the $\angle$BMC = the $\angle$EHF;

$\therefore$ the pt. C coincides with the pt. F;

$\therefore$ the arc BKC = the arc ELF.

But the circumferences are equal;

$\therefore$ the remaining arc BAC = the remaining arc EDF. Q.E.D.

EXERCISES.

1. If two opposite sides of a cyclic quadrilateral are parallel, the other two sides are equal, and the diagonals are equal. XXXI. 1.

2. AB, CD are parallel chords of a circle, and E is the middle point of the arc BD; AE meets CD produced in F. Prove that the triangle CEF is isosceles. XXXI. 6.

3. Two given circles intersect in the points A and B. Any line is drawn through A cutting the circles again in the points C and D respectively. Prove that the angle CBD is constant. XXXI. 8.

4. The opposite sides AB, CD of a cyclic quadrilateral ABCD are equal. Show that the diagonals AC, BD are equal, and the opposite sides AD, CB are parallel. XXXI. 11.

5. A triangle, right-angled at C, is described on a given base AB: find the locus of C. XXXII. 2.

6. AB and CD are two perpendicular chords of a circle: show that the sum of the arcs AC and BD is equal to half the circumference of the circle. XXXII. 3.

7. If one circle be described on the radius of another as diameter, any chord of the latter drawn from the point in which the circles meet is bisected by the former. XXXII. 13.

8. Two circles cut one another in A, B; AC, AD are diameters. Prove that CD passes through B. XXXII. 5.

9. The circle described with centre A and radius AB cuts the circle circumscribing the rectangle ABCD in the point E. Show that CE is equal to AD, and that DE is parallel to AC. XXXII. 9.

10. P, R, S are the middle points of the sides of the triangle ABC. and AD is perpendicular to BC: prove that the points P, R, S, D are concyclic. XXXII. 10.

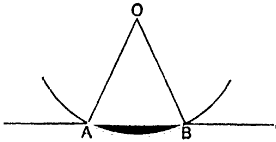
11. AB is the diameter of a semi-circle; D and E are any two points in the circumference. Prove that if the chords joining A and B with D and E each way intersect at F and H, then FH produced is at right angles to AB. XXXII. 11.

12. If semi-circles are described on two sides of a triangle they intersect the base in the same point. XXXIII. 6.

DEFINITION.—A **tangent** to a circle is a straight line which meets the circumference, and, being produced, does not cut it.

PROPOSITION 14. THEOREM.

The straight line drawn perpendicular to a radius of a circle at its extremity is a tangent to the circle.



In a circle, centre O, AB is drawn perp^r to the radius OA.

It is reqd. to prove that AB *is a tangent to the circle.*

Suppose that AB cuts the circle again at the pt. B. Join OB.

OA = OB; $\therefore$ $\angle$OBA = $\angle$OAB = a rt. $\angle$, *Given.*

i.e. the $\triangle$OAB has two right $\angle$s, which is impossible;

$\therefore$ AB cannot cut the circle again,

i.e. AB is a tangent to the circle. Q.E.D.

COR. 1. At the pt. A only one line can be drawn perpendicular to OA, and that line is a tangent.

$\therefore$ **A tangent to a circle is perpendicular to the radius drawn from the point of contact.**

COR. 2. For the same reason as in Cor. 1:

At every point on a circle one, and only one, tangent to the circle can be drawn.

Cor. 3. If A is the point of contact of the tangent AB, only one line can be drawn perp^r to it at the pt. A, and that perp^r is a radius of the circle.

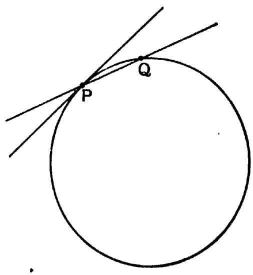
∴ **The perpendicular to a tangent at its point of contact passes through the centre.**

Alternative Definition.—*If the two points in which a secant cuts the circumference of a circle move up to one another, the secant in its ultimate position when those points coincide is the tangent to the circle at the point of coincidence.*

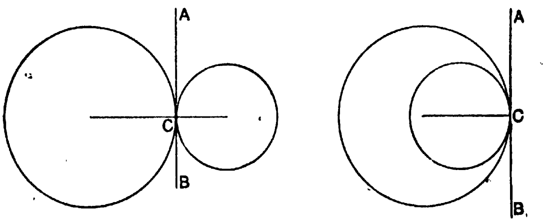
Thus if a straight line, drawn through a point P on the circumference of a circle, meet the curve again at Q, and if the straight line be turned round the point P until the point Q approaches indefinitely near to P, the straight line in its ultimate position is the tangent to the circle at P.

We may therefore look upon a tangent as a straight line passing through two points on a circle indefinitely near to one another.

Also we see that a tangent meets a circle, but does not cut it.



In the same way, two circles touch one another when their points of intersection move up to one another and coincide.



These figures show a tangent touching each circle at C. The circles also touch one another, and ACB is a common tangent in each case.

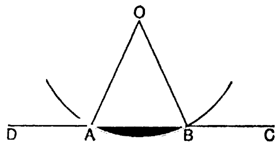
Proposition 14. Theorem. (Alternative Proof.)

The straight line drawn perpendicular to a radius of a circle at its extremity is a tangent to the circle.

Take two points A and B, on the circumference of a circle of centre O.

Join AB, and produce it both ways to C and D.

Draw the radii OA, OB.



$$OA = OB;$$

$\therefore$ the $\angle OAB =$ the $\angle OBA$. (I. 9.)

$\therefore$ their supplements are equal.

i.e. the $\angle OAD =$ the $\angle OBC$.

But when the points A and B move up to one another, the radii OA, OB coincide.

$\therefore$ The $\angle$s OAD, OBC become adjacent $\angle$s, and therefore right angles.

Q.E.D.

Cor. 1. At the point A only one line can be drawn perpendicular to OA, and that line is a tangent.

$\therefore$ **A tangent to a circle is perpendicular to the radius drawn from its point of contact.**

Cor. 2. For the same reason as in Cor. 1:

At every point on a circle one, and only one, tangent to the circle can be drawn.

Cor. 3. If A is the point of contact of the tangent AB, only one line can be drawn perp^r^ to it at the pt. A, and that perp^r^ is a radius of the circle.

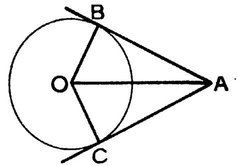
$\therefore$ **The perpendicular to a tangent at its point of contact passes through the centre.**

Cor. 4. When the points of intersection of two circles move up to one another and coincide, the two circles touch each other, and the common chord produced becomes the common tangent at the point of contact of the circles.

Proposition 15. Theorem.

The two tangents drawn to a circle from an external point are equal to one another.

Let AB, AC be two tangents drawn to the circle whose centre is at O.



It is reqd. to prove that AB = AC.

Join OA, OB, OC.

AB and AC are tangents;

∴ the ∠s B and C are rt. ∠s. (III. 14.)

Hence in the rt.-angled △s OBA, OCA,

the hypotenuse OA is common to both,

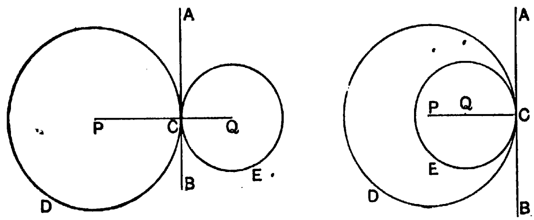
and OB = OC (radii);

∴ AB = AC. (I. 14.)

Q.E.D.

Proposition 16. Theorem.

If two circles touch, either internally or externally, the point of contact lies on the line of centres.



Let the circles, CD, CE, centres P and Q, touch at C, and let ABC be the common tangent at C.

It is reqd. to prove that P, C, Q *are in a str. line.*

AB touches the circle CD at C.

∴ the perp^r to AB at C passes through the centre P.

For the same reason, the perp^r to AB at C passes through the centre Q.

$\therefore$ P, C, Q are in a str. line. Q.E.D.

The converse of this proposition is also true, viz.:

If two circles meet on the line of centres, they touch one another at that point.

Let the circles meet at the pt. C on PQ the line of centres, and let BCA be perp^r to PQ.

CA is a tangent to the circle CD, for it is perp^r to the rad. CP.

Also CA is a tangent to the circle CE, for it is perp^r to the rad. CQ.

$\therefore$ BCA is a common tangent to the two circles at C,

i.e. the circles touch one another at the pt. C.

EXERCISES.

1. Two circles touch internally at A, and ABC is drawn to meet the circles at B and C. Prove that the radii to the points B and C are parallel. XXVIII. 1.

2. Two circles touch externally at A, and BAC is drawn to meet the circles at B and C. Prove that the radii to the points B and C are parallel. XXVIII. 2.

3. From O, whose distance from the centre of a circle is equal to the diameter, tangents OP, OQ are drawn. Show that OPQ is an equilateral triangle. XXVIII. 5.

4. AB is a diameter and AP a chord of a circle; AQ is a chord bisecting the angle BAP. Prove that the tangent at Q is perpendicular to AP. XXVIII. 7.

5. Find the locus of the centres of circles which touch (1) a given straight line, (2) a given circle, at a given point. XXVIII. 8.

6. Two tangents to a circle make equal angles with the chord joining the points of contact. XXIX. 5.

7. If TP, TR are tangents to a circle whose centre is O, TO bisects the angle POR. XXIX. 6.

8. The sums of the opposite sides of a quadrilateral circumscribing a circle are equal. XXIX. 8.

9. One circle is entirely within another: draw the greatest and least chords of the outer which touch the inner. XXIX. 10.

10. Through a given point within a given circle draw two equal chords containing a right angle. XXIX. 11.

11. If a tangent to two circles which touch externally at C meet them in A and B, then ACB is a right angle. XXIX. 13.

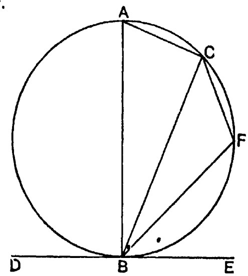
12. If a straight line touch a circle and be parallel to a chord, the point of contact is the mid. point of the arc cut off by the chord.

13. All straight lines which cut off equal arcs from a given circle touch another circle.

14. The perpendicular drawn from the mid. point of an arc to the chord of the arc is equal to the perpendicular drawn from the same point to the tangent at an end of the arc.

PROPOSITION 17. THEOREM.

If a straight line touch a circle, and from the point of contact a chord be drawn, the angles between the chord and tangent are equal to the angles in the alternate segments.



Let ABFC be any circle, DBE the tangent at B, and BC any chord drawn from B.

It is reqd. to prove that the $\angle$ EBC *= the* $\angle$ *in the segment* BAC, *and the* $\angle$ DBC *= the* $\angle$ *in the segment* BFC.

Let BA be a diameter.

Join AC, CF, FB.

AB is a diameter ;

∴ the $\angle$ACB is a rt. $\angle$ (angle in a semi-circle) ; (III. 7.)

∴ the $\angle$CBA + the $\angle$BAC = a rt. $\angle$ (I. 7.)

= the $\angle$EBA (since EB is a tangent). (III. 14.)

Take away the common $\angle$CBA ;

∴ the $\angle$EBC = the $\angle$BAC.

Now ABFC is a cyclic quadrilateral ;

∴ the $\angle$BFC is the supplement of the $\angle$BAC. (III. 9.)

Also the $\angle$DBC is the supplement of the $\angle$EBC ; (I. 1.)

∴ the $\angle$DBC = the $\angle$BFC. Q.E.D.

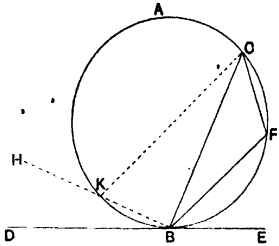
Proposition 17 *may also be proved by regarding a tangent as the limiting position of a secant.*

Let a str. line HKB cut the circle at K. Join KC.

The ∠HKC
= the supplement of the ∠BKC (I. 1.)
= the ∠BFC, since BFCK is a cyclic quad^l.

This remains true while the line HB turns round B into the position of the tangent DB.

The ∠HKC has then become the ∠DBC;



∴ the ∠DBC = the ∠BFC;

∴ also the supplement of the ∠DBC = supplement of the ∠BFC.

But the supplement of the ∠DBC = the ∠EBC, (I. 1.)

and the supplement of the ∠BFC = the ∠BAC in the cyclic quad^l BACF.

∴ the ∠EBC = the ∠BAC. Q.E D.

EXERCISES.

1. If the tangent PT at any point P on a circle meet a diameter AB produced at T, and if P be joined to B, the angle ATP together with twice the angle BPT is equal to a right angle.

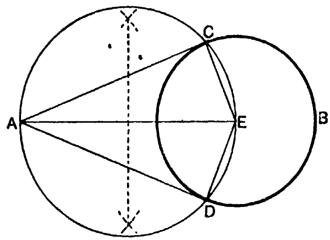
2. A circle is described about an equilateral triangle ABC, and tangents at A and B meet in D; prove that ABD is an equilateral triangle.

3. AB is trisected in the points C, D, and on CD an equilateral triangle CPD is described. Show that a circle passing through the three points B, C, P will touch the straight line AP.

4. If two circles touch externally, and two straight lines be drawn through the point of contact, the chords joining their extremities will be parallel.

PROPOSITION 18. PROBLEM.

To draw a tangent to a circle from a given point.



(i) If the given point is on the circumference, the straight line drawn from it at right angles to the radius through the point is the required tangent. (III. 14.)

(ii) Let A be the given external point, and BCD the given circle.

Take the centre E.

Join AE.

On AE as diameter describe a circle cutting the circle BCD at C, D.

AC, AD shall be tangents to the circle BCD.

EDA, ECA are semi-circles;

$\therefore$ the $\angle$s EDA, ECA are right angles. (III. 7.)

But ED, EC are radii of the circle BCD;

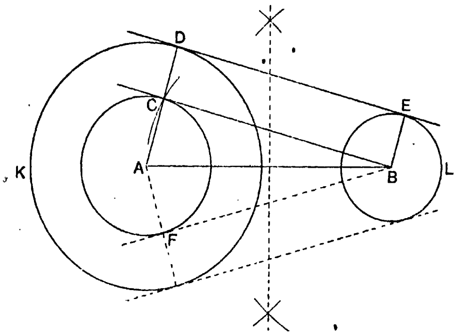
$\therefore$ AD, AC are tangents. (III. 14.)

COROLLARY. Since the right-angled $\triangle$s EDA, ECA have a common hypotenuse and ED equal to EC, it follows that AD = AC; also the $\angle$AEC = the $\angle$AED and the $\angle$EAC = the $\angle$EAD;

i.e. two tangents drawn from an external point to a circle are equal, and subtend equal angles at the centre.

PROPOSITION 19. PROBLEM.

To draw a common tangent to two given circles.



Let A be the centre of the larger circle DK, B the centre of the other EL.

(i) To draw common tangents which do not cut AB.

With centre A, and rad. equal to the *difference* of the radii of the given circles, describe a circle.

From B draw a tangent BC to this circle. (III. 18.)

Join AC, and produce it to meet the larger given circle at D.

Draw BE a rad. || to ACD and on the same side of AB.

Join DE.

ED will be a common tangent.

By construction, CD is equal and || to BE;

$\therefore$ DCBE is a parallelogram. (II. 4.)

Also, the $\angle$ DCB is a rt. $\angle$, for BC is a tangent at C;

$\therefore$ DCBE is a rectangle. *Def.*

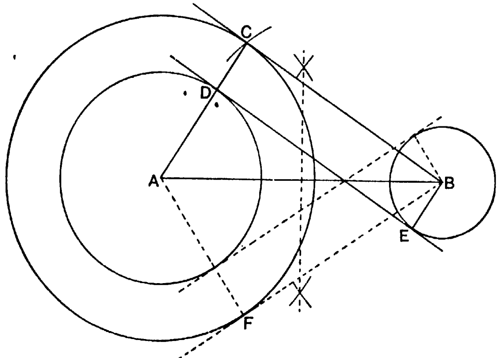
$\therefore$ DE is perp^r to both the radii AD, BE;

$\therefore$ it is a common tangent.

By drawing a tangent BF on the opp. side of BA to BC, we obtain in the same way a second tangent which does not cut AB.

(ii) To draw common tangents which cut AB.

[This of course is impossible if the circles cut one another.]



With centre A (or B) and rad. equal to the *sum* of the radii of the given circles describe a circle, and from B (or A) draw a tangent BC to this circle.

Join AC cutting the given circle at D.

Draw the rad. BE ‖ to ADC, and on the opposite side of AB.

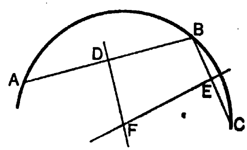
Join DE. DE will be a common tangent.

The proof is the same as in part (i).

Another tangent cutting AB may be obtained by drawing the tangent BF on the opp. side of AB to BC.

PROPOSITION 20. PROBLEM.

To describe a circle of which an arc is given; and to find the centre of a given circle.



1. Let ABC be the given arc.

It is required to complete the circle.

Join AB, BC, and bisect these chords at D, E. (I. 21.)

Draw DF, EF perp^r to AB, BC respectively. (I. 22.)

F shall be the centre of the required circle.

AB is a chord, and DF bisects it at right angles;

∴ the centre lies in DF. (III. 1.)

Similarly the centre lies in EF;

∴ if with F as centre and FA as radius we describe a circle, it will be the circle required.

2. Let ABC be a complete circle whose centre is required.

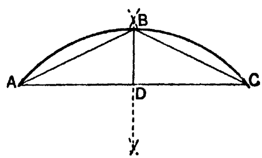
Bisect a chord BC at rt. ∠s by a line meeting the circumference at H and K.

The centre must lie in HK; (III. 1.)

∴ it lies at the middle point of HK.

PROPOSITION 21. PROBLEM.

To bisect a given arc.



Let ABC be the given arc.

It is required to bisect it.

Join AC.

Bisect AC at D. (I. 21.)

From D draw DB perp^r to AC to meet the arc at B. (I. 22.)

The arc ABC shall be bisected at B.

Join AB, BC.

In the △s ADB, CDB, *Cons.*

(1) AD = CD,

(2) DB is common,

(3) the ∠ADB = the ∠CDB (rt. angles);

∴ AB = CB. (I. 9.)

Now the arcs AB, BC are both minor arcs, for BD produced is a diameter.

Also they are cut off by equal chords;

∴ the arc AB = the arc BC; (III. 13.)

i.e. arc ABC is bisected. Q.E.F.

PROPOSITION 22. PROBLEM.

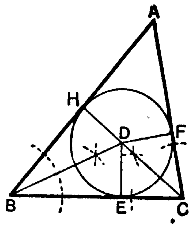
To inscribe a circle in a given triangle.

Let ABC be the given △.

It is required to inscribe a circle in the △ABC.

Bisect the ∠s B and C by BD, CD. (I. 20.)

Draw DE, DF, DH perp^r to BC, CA, AB respectively. (I. 24.)



In the △s DBE, DBH,

(1) the ∠DBE = the ∠DBH, *Cons.*

(2) the rt. ∠DEB = the rt. ∠DHB,

(3) DB is common;

∴ DE = DH. (I. 10.)

Similarly DE = DF.

∴ the circle described with centre D and radius DE will pass through E, F, H, and will touch the sides of the △ABC, because the angles at E, F, H are rt. angles. (III. 14.)

This problem might be stated thus—*To describe a circle to touch three given straight lines which intersect in three points.* It would then be obvious that four circles might be drawn. In addition to the inscribed circle there would be three circles each touching one side of the △ and the other two produced. These are **escribed** circles.

COROLLARY. If a, b, c, r be the lengths of BC, CA, AB, DE, $\frac{1}{2}ar$ = △DBC.

$$\triangle ABC = \triangle DBC + \triangle DCA + \triangle DAB = \tfrac{1}{2}ar + \tfrac{1}{2}br + \tfrac{1}{2}cr$$

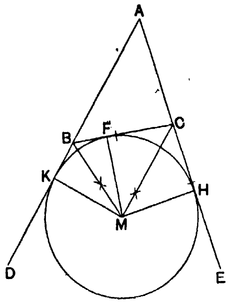
$$= \tfrac{1}{2}(a+b+c)r = sr, \text{ where } s = \text{the semi-perimeter.}$$

∴ r = *area of the △ ÷ the semi-perimeter.*

Exercise. Find the centre and calculate the radius of the in-circle of a △ whose sides are 5, 12, 13 cm.

PROPOSITION 23. PROBLEM.

To draw an escribed circle of a triangle.



Let ABC be the given △.

It is required to draw a circle which shall touch BC *and the produced parts of* AB *and* AC.

Produce AB to D, and AC to E.

Bisect the ∠s DBC, BCE by str. lines meeting at M.

Draw MF, MH, MK perp^r to BC, CE, BD respectively.

In the △s KBM, FBM,

(1) the ∠KBM = the ∠FBM, *Cons.*

(2) the rt. ∠K = the rt. ∠F

(3) BM is common ;

∴ MK = MF. (I. 10.)

Similarly MF = MH.

∴ a circle with centre M and radius MF will pass through F, H, and K, and will touch the sides because the ∠s at F, H, and K are rt. angles.

Similarly the other two escribed circles can be drawn.

NOTE.—BK = BF, since two tangents drawn from an external point to a circle are equal ;

∴ AK = AB + BF.

Similarly AH = AC + FC ;

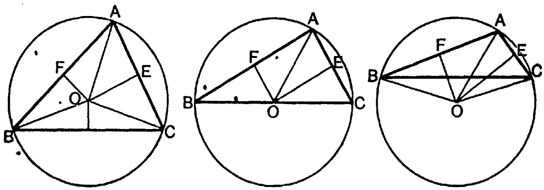
∴ AK + AH = AB + AC + BC = perimeter of the △.

But AK = AH, since they are tangents ;

∴ each = the semi-perimeter (*s*).

PROPOSITION. 24 PROBLEM.

To describe a circle about a given triangle.



Let ABC be the given $\triangle$.

It is reqd. to describe a circle about the $\triangle$ABC.

Bisect AC, AB at E and F. (I. 21.)

From E, F draw perp^{rs} to AC, AB meeting in O. (I. 22.)

Join AO, BO, CO.

In the $\triangle$s AEO, CEO,

(1) AE = CE, *Cons.*

(2) EO is common,

(3) the rt. $\angle$AEO = the rt. $\angle$CEO;

$\therefore$ AO = CO. (I. 9.)

Similarly AO = BO.

$\therefore$ a circle described with centre O and radius OA will pass through B and C. Q.E.F.

NOTES.—**The same construction is required to describe a circle through three given points.**

If B and C be joined to any point P in the arc BC opposite to A, a $\triangle$PBC is formed whose vertical $\angle$ is the supplement of A, and whose circum-circle is the same as that of the $\triangle$ABC.

Hence *triangles whose bases are equal and whose vertical angles are equal or supplementary have equal circum-circles.*

EXERCISES.

1. In an equilateral triangle the in-centre is equidistant from the vertices. XXXV. 1.

2. A side of an equilateral triangle described about a circle is double of a side of the inscribed equilateral triangle. XXXV. 2.

3. Two equiangular triangles ABC, DEF are inscribed in the same circle; prove that they are equal in all respects. XXXV. 3.

4. ABC is a triangle, AP a diameter of its circum-circle, and BE, CF the perpendiculars upon AC, and AB respectively meet in T; prove that BPCT is a parallelogram. XXXV. 5.

5. D is the in-centre of the triangle ABC; E is the centre of the circle touching AB, AC produced and also BC. Show that DE is a diameter of a circle through B and C. XXXV. 6.

6. In a right-angled triangle, the sum of the sides containing the right angle is equal to the sum of the diameters of the in-circle and circum-circle. XXXV. 7.

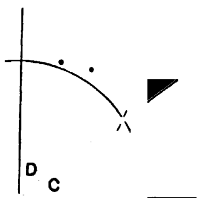
7. O is the intersection of perpendiculars AD, BE, CF from the vertices of the triangle ABC to the opposite sides. Prove that O is the centre of the circle inscribed in the triangle DEF. XXXV. 8.

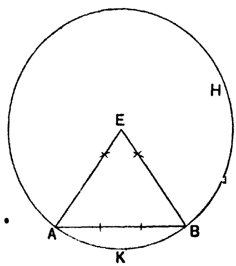
8. The external bisectors of the angles ABC, ACB of a triangle meet in D. Show that AD passes through the centre of the in-circle.
XXXV. 9.

9. AB is the side of an equilateral triangle inscribed in a circle whose centre is O. AO meets the circle in D. Prove that the triangle BOD is equilateral. XXXV. 10.

Proposition 25. Problem.

On a given straight line to construct a segment of a circle containing an angle equal to a given angle.





Let AB be the given str. line, and C the given ∠.

It is reqd. to construct on AB *a segment containing an angle equal to* C.

At A and B make the ∠s BAE and ABE each equal to $90° \sim C$.

Because these ∠s are equal, AE = BE.

With centre E and radius EA or EB describe a circle AKBH.

(1) If C is an acute $\angle$, AHB is the required segment.

For the $\angle$EAB + the $\angle$EBA = 90° − C + 90° − C = 180° − 2C.

But, in the $\triangle$EAB,

$$\text{the } \angle EAB + \text{the } \angle EBA = 180° - E.$$

$$\therefore\ 2C = E;$$

$$\therefore\ \text{the } \angle C = \tfrac{1}{2} \text{ the } \angle E = \text{the } \angle AHB. \qquad \text{(III. 4.)}$$

(2) If C is an obtuse $\angle$, AKB is the required segment.

For the $\angle$EAB + the $\angle$EBA = C − 90° + C − 90° = 2C − 180°.

$$\text{But the } \angle EAB + \text{the } \angle EBA = 180° - E;$$

$$\therefore\ E = 360° - 2C.$$

$$\text{The } \angle AHB = \tfrac{1}{2}E = 180° - C;$$

$$\therefore\ C = 180° - \text{the } \angle AHB = \text{the } \angle AKB.$$

[The proof is simpler when the angle is given in degrees.]

NOTE.—When C is a rt. $\angle$, describe a semi-circle on AB.

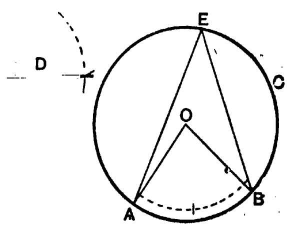
Exercise. On a base of 4 cm. describe a segment containing an $\angle$ of 54°.

PROPOSITION 26. PROBLEM.

From a given circle to cut off a segment containing an angle equal to a given angle.

Let ABC be the given circle, and D the given angle.

It is required to cut off from the circle a segment containing an angle equal to D.



Take O the centre of the circle. (III. 20.)

Join OA, and make the $\angle$AOB equal to twice the $\angle$D.

ACB will be the reqd. segment.

Draw AEB, any angle in this segment.

Then the $\angle$AEB at the circumference

$=$half the $\angle$AOB at the centre (III. 4.)

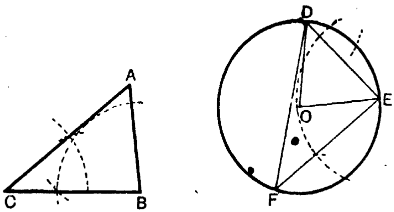
$=$the $\angle$D;

$\therefore$ ACB is the reqd. segment.

Exercise. From a circle of 1 inch radius cut off a segment containing an $\angle$ of 35°.

PROPOSITION 27. PROBLEM.

To inscribe in a given circle a triangle equiangular to a given triangle.



Let ABC be the given $\triangle$, DEF the given circle.

Draw any radius OD, and make the $\angle$DOE equal to twice the $\angle$ACB.

Join DE.

Make the $\angle$DEF equal to the $\angle$ABC;

Join DF.

The $\angle$DFE $=$ half the $\angle$DOE (III. 4.)

$=$the $\angle$C. *Cons.*

The $\angle$DEF $=$ $\angle$B; *Cons.*

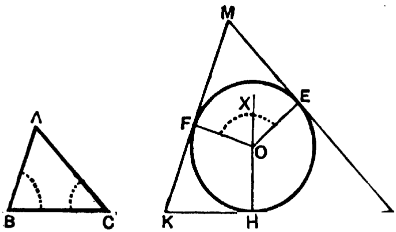
$\therefore$ the $\angle$FDE $=$the $\angle$A; (I. 7.)

$\therefore$ DEF is the $\triangle$ reqd. Q.E.F.

Exercise. In a circle of 1·2 in. radius inscribe a $\triangle$ with two of its $\angle$s 45° and 50°.

PROPOSITION 28. PROBLEM.

To describe about a given circle a triangle equiangular to a given triangle.



Let ABC be the given △, HEF the given circle.

It is required to describe about HEF *a △ equiangular to* ABC.

Take O the centre of the circle HEF.

Draw any radius OH.

Produce HO to X.

Draw radii OE, OF making the ∠XOE equal to the ∠ACB and the ∠XOF equal to the ∠ABC.

Draw tangents at H, E, F, forming the △KLM. (III. 14.)

KLM shall be the required △.

The angles of the quad[l] OHLE are together equal to 4 rt. ∠s.

But the ∠s at H and E are rt. ∠s;

∴ the ∠HLE = the supplement of the ∠HOE = the ∠XOE

= the ∠ACB. *Cons.*

Similarly the ∠HKF = the ∠ABC;

∴ the ∠M = the ∠A. (I. 7.)

∴ KLM is the △ req. Q.E.F.

Exercise. About a circle of 3 cm. radius describe a △ containing ∠s of 50° and 75°.

Proposition 29. Problem.

To circumscribe a square about a given circle.

Draw two diameters at rt. ∠s. At their extremities draw tangents to the circle.

These four tangents will form a square.

(The proof is left to the student.)

Proposition 30. Problem.

To inscribe a square in a given circle.

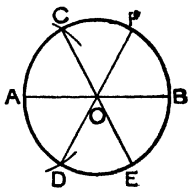
Draw two diameters at rt. ∠s.

The four str. lines joining their extremities will form a square.

(The proof is left to the student.)

Proposition 31. Problem.

To circumscribe a regular hexagon about a given circle.



If O is the centre of the given circle, draw any diameter AOB.

With centre A and rad. AO describe a circle cutting the given circle at C and D.

Draw the diameters COE, DOF.

The tangents at A, D, E, B, F, C will form a regular hexagon.

(The proof is left to the student.)

Proposition 32. Problem.

To inscribe a regular hexagon in a given circle.

In the construction of the previous problem,

ADEBFC will be a regular hexagon.

PROPOSITION 33. PROBLEM.

To circumscribe a regular octagon about a given circle.

Draw two diameters at rt. ∠s. Draw two more diameters bisecting the angles between the first two.

The fig. formed by drawing tangents at the extremities of these four diameters will form a regular octagon.

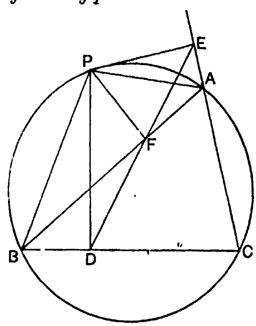
PROPOSITION 34. PROBLEM.

To inscribe a regular octagon in a given circle.

In the construction of the previous problem the lines joining the ends of the diameters, in order, will form a regular octagon.

PROPOSITION 35. SIMSON'S LINE. THEOREM.

The feet of the perpendiculars drawn from any point on the circum-circle of a triangle to the three sides lie in a straight line.



Let ABC be any triangle,

P any point on the circum-circle; and let PD, PE, PF be perp^r to the sides BC, CA, AB.

It is reqd. to prove that D, E, F *are in a str. line.*

Join DF, FE, PA, PB.

PEA, PFA are rt. ∠s;

∴ PEAF is a cyclic quadrilateral.

Also the circle on PB as diameter goes through F, and D since the ∠s PFB, PDB are rt. ∠s.

∴ the ∠PFE = the ∠PAE (in the same segment)

= the supplement of the ∠PAC

= the ∠PBC

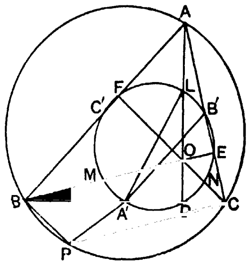
= the supplement of the ∠PFD (in quadrilateral PBDF).

∴ EF and FD are in a str. line. (I. 2.) Q.E.D.

This straight line is sometimes known as the *pedal* line of P, sometimes as the *Simson* line of P

Proposition 36. The Nine-Points Circle. Theorem.

In any triangle the feet of the perpendiculars from the vertices to the sides, the middle points of the lines joining the vertices to the orthocentre, and the middle points of the sides all lie on a circle (called the nine-points circle of the triangle).



Let ABC be any triangle; let AD, BE, CF be the perp^rs to the sides; O the orthocentre; L, M, N the mid. points of AO, BO, CO; A′, B′, C′ the mid. points of BC, CA, AB.

It is reqd. to prove that the pts. D, E, F, L, M, N, A′, B′, C′ *lie on a circle.*

Join A′L, B′L, A′B′.

L, B′ are the mid. points of AO, AC respectively;

∴ LB′ is ∥ to OC. (II. 6.)

B′, A′ are the mid. points of AC, BC respectively;

∴ B′A′ is ∥ to AB. (II. 6.)

But OC and AB are at rt. ∠s;

∴ LB′ and B′A′ are at rt. ∠s.

∴ the circle whose diam. is LA′ goes through B′. (III. 8.)

Similarly it goes through C′.

It also goes through D (since LDA′ is a rt. ∠);

∴ the circle through A′B′C′ goes through D and L;

similarly it must go through E and M and through F and N. Q.E.D.

Corollary. *The diameter of the nine-points circle is half the diameter of the circum-circle.*

Let AP be a diameter of the circum-circle.

Then PC is at rt. ∠s to AC, (III. 7.)

∴ PC is ∥ to BO.

Similarly PB is ∥ to CO;

∴ PBOC is a parallelogram.

Now the diagonals of a par^m bisect each other, and we know A′ is the mid. point of BC;

∴ A′ is the mid. point of OP.

But L is the mid. point of OA;

∴ A′L is half of AP: for the line joining the mid. points of two sides of a triangle is half the base.

EXERCISES.

1. Draw a circle passing through two given points, and having a given radius. XXXIV. 1.

2. Through a given point within a circle draw a chord which shall be bisected by the point. XXXIV. 4.

3. Through a given point draw a chord of a given circle equal to a given length. XXXIV. 5.

4. Draw a tangent to a given circle parallel to a given straight line. XXXIV. 7.

5. Draw a tangent to a given circle perpendicular to a given straight line. XXXIV. 8.

6. Through a given point within a given circle draw the shortest chord. XXXIV. 9.

7. In a given circle draw a chord parallel to a given straight line and equal to a given length. XXXIV. 13.

8. Describe a circle touching three given straight lines, of which two are parallel. XXXIV. 14.

9. Describe a circle of given radius to touch two given intersecting straight lines. XXXIV. 15.

10. With a given centre describe a circle to cut off a given length from a given straight line. XXXIV. 16.

11. Inscribe an equilateral triangle in a given circle. XXXIV. 17.

12. Divide the circumference of a circle into twelve equal parts. XXXIV. 19.

13. Divide a given circle into two parts so that the angle in one segment shall be double of the angle in the other. XXXIV. 34.

14. Divide a given circle into two parts so that the angle in one segment shall be five times the angle in the other. XXXIV. 35.

15. In a given straight line find a point such that the lines drawn from it to two given points shall contain a right angle. XXXIV. 36.

16. Find a point such that the tangents drawn from it to a given circle may contain a given angle. What is the locus of all such points? XXXIV. 39.

17. Draw a chord in a circle, so that it may be double of its distance from the centre. XXXIV. 40.

18. A circle cuts off equal chords from the sides of a given triangle. Find its centre. XXXIV. 31.

19. Describe a triangle, having given the base, the altitude, and the vertical angle. XXXIV. 20.

20. Draw a diagram showing how far a football must be taken out from a given point on the goal line, in a direction at right angles to the goal line, that the goal may subtend a given angle at the point where the ball is placed.

Show where the goal will subtend the maximum angle. XXXIV. 21.

21. Given the altitude, the vertical angle, and the perimeter of a triangle, construct it. XXXIV. 23.

22. Construct a triangle, having given the vertical angle, one of the sides containing it, and the altitude. XXXIV. 24.

23. Describe a circle of given radius which shall touch two given circles. XXXIV. 26.

24. Describe a circle to pass through a given point and touch a given circle at a given point. XXXIV. 27.

25. Describe a circle which shall pass through a given point and touch a given straight line at a given point. XXXIV. 28.

26. Through one of the common points of two circles, draw a straight line terminated by the circumferences and bisected at the common point. XXXIV. 29.

27. Draw the common tangents to two circles whose radii are 2 and 3 cm., and whose centres are 8 cm. apart.

28. Describe a circle to touch a given circle, and to touch a given straight line at a given point. XXXVI. 2.

(Take BOC the diameter of the given circle at rt. $\angle$s to the given str. line. Join B (or C) to the given pt. A, meeting the circle at D. The centre of the circle reqd. will lie in OD produced. The centre also lies in the perpendicular to the given line at A.)

29. The perpendiculars drawn from the vertices of a triangle to the opposite sides meet in a point. XXXVI. 3.

(Proof by means of Prop. III. 5, 6.)

(This point is called the **orthocentre** of the triangle.)

(Draw the perpendiculars AD, BE, meeting at O. Join CO and produce it to meet AB at F. The pts. A, B, D, E are concyclic and also the pts. D, C, E, O. Then

$$\angle FAO + \angle FOA = \angle BED + \angle COD = \angle BED + \angle DEC = \text{a rt. } \angle.$$

Hence, etc.)

(The $\triangle$ DEF is called the pedal or orthocentric triangle.)

30. In an acute-angled triangle, the perpendiculars from the vertices of a triangle on the opposite sides bisect the angles of the pedal triangle. XXXVI. 4.

31. Given the base and vertical angle of a triangle, prove that the locus of the intersection of its medians is an arc of a circle. XXXVI. 5.

(If D, E be the points of trisection of the base, DE will be found to be the chord of the arc.)

32. All circles whose centres lie on a given straight line, and which pass through a fixed point, pass through a second fixed point. XXXVI. 6.

33. The in-circle of the triangle ABC touches BC at D, AC at E, and AB at F. Prove that

$$AF = AE = s - a, \quad BF = BD = s - b, \quad CD = CE = s - c,$$

where a, b, c denote the sides BC, AC, AB respectively, and $2s$ denotes the sum of the sides. (AF + BD + CD = s, *i.e.* AF + a = s.) XXXVI. 7.

34. If an escribed circle of the triangle ABC touches BC externally at D, prove that, with the above notation, BD = $s - c$ and CD = $s - b$. XXXVI. 9.

35. If T is the orthocentre of the triangle ABC, and AP is a diameter of its circum-circle, prove that PT and BC bisect one another. (Prove that BPCT is a par^{m}.) XXXVI. 10.

36. From O, the centre of the circum-circle of the triangle ABC, OL is drawn at right angles to BC. If T is the orthocentre of the triangle, prove that AT = 2OL. (As in 42 prove that BPCT is a par^{m}. Hence L is the mid. pt. of PT.) XXXVI. 11.

37. Given the base and vertical angle of a triangle, find the locus of its orthocentre. XXXVI. 12.

EXERCISES.

Loci.

1. What is the locus of the centres of all circles which touch a given straight line at a given point? XXXVII. 1.

2. Equal chords are drawn in a circle: what is the locus of their middle points? XXXVII. 2.

3. Tangents are drawn from a fixed point to a series of concentric circles: find the locus of their points of contact. XXXVII. 3.

4. Find the locus of the middle points of chords of a circle which pass through a fixed point. XXXVII. 4.

5. ABC is a triangle of given base BC and given vertical angle A. BA is produced to D so that AD = AB: find the locus of D. XXXVII. 5.

6. Two circles touch a given straight line at given points, and also touch each other: find the locus of their point of contact. XXXVII. 6.

7. A and B are two fixed points on a circle, and CD is any diameter. Find the locus of the point of intersection of AD and BC. XXXVII. 7.

8. Given the base and the vertical angle of a triangle, find the locus of the centre of its circum-circle. XXXVII. 8.

9. Given the base and vertical angle of a triangle, find the locus of the centre of its inscribed circle. XXXVII. 9.

10. Given the base and vertical angle of a triangle, find the locus of the centre of that escribed circle which touches the base externally. XXXVII. 10.

11. AB is a fixed chord of a circle, and C any other point on the circumference; if the parallelogram BADC be completed, find the locus of the intersection of its diagonals. XXXVII. 11.

12. A straight ruler slides with its ends on two fixed straight lines at right angles to one another; find the locus of its middle point. XXXVII. 12.

13. AB and CD are drawn parallel to one another on the same side of the straight line AC: find the locus of the point of intersection of the bisectors of the angles BAC, ACD; A and C being fixed points. XXXVII. 13.

14. Two circles, centres C and D, cut one another at A and B, and PAQ is any straight line through A terminated by the circumferences: prove that the locus of the point of intersection of PC and QD is a circle passing through B. XXXVII. 14.

15. A is a fixed point on the circumference of a circle, and AB is any chord. On AB is described a segment of a circle containing a given angle: prove that the locus of its centre is one or other of two fixed circles. XXXVII. 15.

EXERCISES.

Numerical Examples in Circles, Chords and Tangents.

1. From an external point T a tangent TP is drawn to a circle whose centre is O. If TO, TP measure 26 and 24 inches respectively, find the length of the radius. XXXVIII. 1.

2. O being the centre of a circle, find the length of the tangent TP if TO be 37 inches, and radius = 35 inches. XXXVIII. 2.

3. Two tangents make an angle 120° with each other, and the chord joining the points of contact is 6 inches long. Find the radius. XXXVIII. 3.

4. In a circle whose radius is 65 inches a chord is 126 inches: find its distance from the centre. XXXVIII. 4.

5. In a circle of radius 85 inches there are two parallel chords whose lengths are 72 inches and 102 inches; find their distance apart. XXXVIII. 5.

[The **height** of an arc is the line drawn to the arc from the mid. point of the chord perpendicular to the chord.]

6. Find the chord of an arc whose height is 8 inches in a circle of radius 13 inches. XXXVIII. 6.

7. In a circle of diameter 54 decimetres the height of an arc is 15 decimetres; find the chord of the arc, and the chord of half the arc. XXXVIII. 7.

8. The chord of an arc is 64 feet and its height is 8 feet; find the radius. XXXVIII. 8.

9. On a line 40 centimetres long and on each half of it are described semi-circles. Prove that the radius of the circle inscribed in the space enclosed by the three semi-circles is $6\frac{2}{3}$ centimetres. XXXVIII. 9.

10. Two equal circles of 25 centimetres radius have their centres 48 cm. apart. Find the length of their common chord. XXXVIII. 10.

11. Describe a circle of radius 5 cm., draw a chord of length 6 cm., and describe a concentric circle touching this chord. Measure the radius of this circle. XXXVIII. 11.

12. With centres 5 cm. apart, draw circles of radii 3 and 4 cm., and measure their common chord. XXXVIII. 12.

13. In a circle of 70 inches diameter, a chord is drawn 56 in. in length; find its distance from the centre. XXXVIII. 13.

14. In a circle of radius 70 in. a chord 48 in. long is drawn: find its distance from the centre correct to two decimal places of an inch. XXXVIII. 14.

15. The chord of an arc is 10 cm. long, and its middle point is 5 cm. from the middle point of the chord: find the radius of the circle. XXXVIII. 15.

16. Describe a circle of radius 2·6 cm. passing through two points 4·8 cm. apart. Describe your construction, and find the distance of the chord from the centre. XXXVIII. 16.

17. In a circle of radius 5 cm. two chords of length 6 and 8 cm. are drawn parallel to one another: find their distance apart. XXXVIII. 17.

18. Make a triangle whose sides are 5, 6 and 7 cm. long. Erect perpendiculars at the middle points of two sides, and measure the distance of their point of intersection from the three angular points. What do you deduce from your results? XXXVIII. 18.

19. In a circle of radius 2·7 cm. draw a chord 3·4 cm. long, and measure its distance from the centre. XXXVIII. 19.

20. In a circle of radius 3·5 cm. draw a chord at a distance 2 cm. from the centre, and measure its length. XXXVIII. 20.

21. Draw a circle with a radius of 8 cm. (or if you prefer it, 3 inches) and from a point P distant 16 cm. (or 6 inches) from the centre O of the circle draw a tangent to the circle; measure and state its length. XXXVIII. 21.

22. With your instruments draw a circle of radius 5 cm. (or, if you prefer it, 2 inches), and inscribe in it a regular hexagon. State your construction and prove that all the angles of the hexagon are equal. XXXVIII. 22.

23. A circle is described so as to pass through the points A and B of a square ABCD, and the middle point of CD. Making the sides of the square 5 cm. long, draw the circle. If it cuts AD in F, measure the length of AF as accurately as you can. XXXVIII. 23.

24. The length of each of two tangents drawn from a point to a circle is a, and the length of the chord of contact b: find the radius. XXXVIII. 24.

25. Two tangents drawn from a point to a circle are each 4 cm. long, and the angle between them is 65 degrees; draw the tangents and the circle, and measure the radius as accurately as you can. XXXVIII. 25.

26. ODA, DQC are at right angles. OQB is a straight line. $DA = a$ $DC = b$, $DQ = c$. O is the centre of a circular arc AB, Q the centre of a circular arc BC. Find OA, QC in terms of a, b, c. Find the lengths of OA and QC when $a = 3$, $b = 5$, $c = 3$. XXXVIII. 26.

27. It is required to fix the position of a certain tree on a moor by taking measurements of lengths or angles with reference to two stone pillars. Give as many sets of measurements as you can, each set of which would be sufficient for the purpose. XXXVIII. 27.

28. Draw a straight line BC 8 cm. long, and from B draw a straight line BA, making 60° with BC. If this figure is to be completed into a triangle ABC, what is the greatest possible size of the angle C? What is the least possible size of AC? Give your reasons. XXXVIII. 28.

29. Draw a circle of radius 3 inches, and in it place a chord AB 3·4 inches long. Take four points P, Q, R, S in the greater part of the circumference. Measure and compare the angles APB, AQB, ARB, ASB. What property of a circle is suggested? Prove this property in the most general form you know. XXXVIII. 29.

30. AB is a diameter of a circle, P a point on it, PQ perpendicular to AB and meeting the circumference at Q. For what position of P is PQ greatest? State a relation between the lengths PA, PQ, PB.

If 2000 yards of fencing is to enclose as great a rectangular field as possible, what must be the lengths of the sides? XXXVIII. 30.

31. With instruments make a square equal in area to a rectangle measuring 11 cm. by 7 cm. State construction. Hence find the square root of 77. XXXVIII. 31.

32. On a line AB, 5 cm. in length, describe a segment of a circle containing an angle of 30°. State your construction. If O be the centre of this circle, what is the magnitude of the angle AOB ? XXXVIII. 32.

33. Make a circle 8 cm. in radius, and in it lay in succession 4 chords, AB = 11 cm., BC = 10 cm., CD = 9 cm., DE = 8 cm. By measurement find in degrees $\angle$ABC + $\angle$AEC, $\angle$ABD + $\angle$AED, $\angle$BCD + $\angle$BED, and $\angle$BCE + $\angle$BAE. XXXVIII. 33.

34. Suppose a circle of diameter $2\frac{1}{2}$ inches drawn, and any diameter AB produced to P, so that PB is 2 inches. What would be the length of a tangent from P to the circle ? Explain how you find it. XXXVIII. 34.

35. Take a straight line 5 cm. long. Describe a parallelogram on it, having one angle 45° and an area of 40 sq. cm. Measure one of its longest sides. XXXVIII. 35.

36. If I walk 10 kilometres east and then 4 kilometres north-east, how far am I from my starting-point ? XXXVIII. 36.

37. Draw a circle of radius 3 inches, and in it place chords of 1, 2, 4 inches. Ascertain from your figure which is the nearest to the centre and which farthest from it. XXXVIII. 37.

38. Describe a circle of 2 inches radius, centre O. In it place a chord AB 3 in. long. In the greater of the arcs into which the circle is divided take points P, Q, R. Measure $\angle$s AOB, APB, AQB, ARB ; state the relation between them, and prove your statement. XXXVIII. 38.

39. A straight line PQ cuts a circle in A and B. If E moves along the circle towards A, so that the line PQ turns round A, what happens to PQ as B reaches A ? XXXVIII. 39.

40. O is the centre of a circle, and AB a chord. Calculate the values of the $\angle$OAB when the $\angle$BOA has values 30°, 20°, 10°, 5°, 1°, 30′, 1′. Deduce a rule for drawing the tangent at A. XXXVIII. 40.

41. Draw AB, AC at right angles, one 2 inches and the other 3 inches long. Find a point at which each subtends an angle of 120°. XXXVIII. 41.

42. In a circle of 1 inch radius place a chord of ·8 inch parallel to a given straight line. XXXVIII. 42.

43. Draw all the circles of 0·7 inch radius which touch two given intersecting straight lines. XXXVIII. 43.

44. Draw two circles of ·8 and 1·6 inches radius and with centres 2·8 inches apart. Draw a straight line so that the intercepted chords may be each 1 inch. How many solutions are there? XXXVIII. 44.

45. In a circle of 3 cm. radius inscribe the greatest possible triangle, and find the length of its perimeter. XXXVIII. 45.

RATIO.

If one quantity, a, is three times another quantity, b, of the same kind, a is said to be to b in the ratio of 3 to 1.

This is expressed thus: $a : b :: 3 : 1$, or more usefully thus:

$$\frac{a}{b} = \frac{3}{1}.$$

In the same way, if one quantity is three-quarters of another of the same kind, the first quantity is said to be to the second in the ratio of 3 to 4, and we may express this thus, where a and b are the two quantities:

$$\frac{a}{b} = \frac{3}{4}.$$

CIRCUMFERENCE OF A CIRCLE.

If we measure the circumference of a circle, we find that its length is approximately $3\frac{1}{7}$ times the length of its diameter.

In other words,

$$\frac{\text{circumference}}{\text{diameter}} = \frac{22}{7} \text{ nearly.}$$

This value of the ratio of the circumference of a circle to its diameter is only approximate. Its actual value is found to be incommensurable; *i.e.* if we measure the diameter in any unit, we cannot express the circumference in an exact multiple of that unit.

The ratio of the circumference of a circle to its diameter is usually denoted by the Greek letter π.

Thus, if r denotes the radius of a circle, its

$$\text{CIRCUMFERENCE} = 2\pi r.$$

A more nearly exact value of π than $\frac{22}{7}$ is 3·1416, but it must be remembered that this again is only an approximation, and that its exact value cannot be obtained.

LENGTH OF AN ARC.

If we divide the circumference of a circle into 360 equal parts, each part will subtend at the centre an angle equal to 1 degree. Hence if l denote the length of one part, and c the length of the whole circumference,

$$\frac{l}{c} = \frac{1}{360},$$

and if x denote the length of the arc which subtends an angle of n degrees at the centre,

$$\frac{x}{c} = \frac{n}{360}; \quad \therefore \quad x = \frac{nc}{360}.$$

EXERCISES.

(*In all Examples take* $\pi = \frac{22}{7}$, *unless something is implied to the contrary.*)

On the Circumference of a Circle.

1. Find approximately the lengths of the circumferences of the circles whose radii are respectively:

(1) 6 inches. (2) 4·5 cm.
(3) 2 ft. 4 in. (4) 1 metre. XXXIX. 1.

2. The circumference of a circle is 77 feet; find its diameter. XXXIX. 2.

3. A wheel 6 ft. in diameter makes 6000 revolutions; through what distance does a point on its rim travel? XXXIX. 3.

4. How many laps to a mile are there in a circular track of 80 yards radius? XXXIX. 4.

5. The driving wheel of a bicycle is 32 in. in diameter: how far has the bicyclist travelled when the wheel has made 2000 revolutions? XXXIX. 5.

6. The driving wheel of a locomotive engine is 7 ft. in diameter, and makes 240 revolutions a minute: at what rate is the engine travelling? XXXIX. 6.

7. Show that the perimeter of a regular hexagon is to the circumference of its circumscribed circle as 3 is to π. XXXIX. 7.

8. The diameter of a circular track is 448 ft. How many revolutions will a bicycle-wheel 32 in. in diameter make in travelling 5 times round it? XXXIX. 8.

(In all Examples take $\pi = \frac{22}{7}$, unless something is implied to the contrary.)

EXERCISES.

On Arcs of Circles.

1. The circumference of a circle is 25 ft.: find the length of an arc which subtends an angle of 60 degrees at the centre. XL. 1.

2. An arc of a circle, 14 inches in diameter, subtends an angle of 54 degrees at the centre: find its length. XL. 2.

3. Two radii of a circle, 21 cm. radius, contain an angle of 75 degrees: find the length of the intercepted arc. XL. 3.

4. AB is a chord of a circle, equal in length to the radius of the circle. If its length is 85 inches, find, to the nearest inch, the length of the arc AB. ($\pi = 3{\cdot}1416$.) XL. 4.

5. A chord of a circle of 5 cm. radius is 8 cm. long; draw the circle and the chord, and measure, as accurately as you can, the height of the smaller arc cut off. XL. 5.

6. Find the length of that part of a railway-curve which subtends an angle of $22\frac{1}{2}$ degrees at the centre of a circle whose radius is 1 mile. XL. 6.

7. Find the distance in miles between any two places on the equator which differ in longitude by $6° \, 18'$, assuming the earth's equatorial diameter to be 7925·6 miles. ($\pi = 3{\cdot}1416$.) XL. 7.

8. Two places on the equator are 300 miles apart. What is their difference in longitude? (Diameter of the earth $= 7925{\cdot}6$ miles, $\pi = 3{\cdot}14159$.) XL. 8.

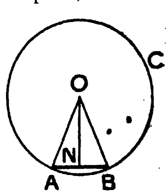
9. The chord of an arc is 16 ft., and the height of the arc 4 ft. What is the radius of the circle? XL. 9.

10. The chord of an arc is 7·2 cm. and the chord of half the arc 3·9 cm.: find the radius of the circle. XL. 10.

AREA OF A CIRCLE.

Let ABC be the circle whose centre is O, and let r be its radius.

Let the circumference be divided into n equal parts, of which AB is any one. Join AB, OA, OB, and draw ON perpendicular to AB. The area of the $\triangle OAB = \frac{1}{2} AB \times ON$.



$\therefore$ the area of the polygon formed by joining all the pts. of division $= \frac{n}{2} AB \, . \, ON$, for the $\triangle$s thus formed are equal in area.

Let the number (n) of divisions be increased indefinitely.

Then *ultimately* each chord AB will coincide with its arc AB.

$\therefore$ the perimeter of the polygon ultimately coincides with the circumference of the circle,

i.e. ultimately n . AB = circumference of circle = $2\pi r$.

Also the perp^r ON ultimately coincides with the radii OA, OB.

$\therefore$ the area of the circle = area of the polygon in the limit

$$= \text{limiting value of } n\,.\,\mathrm{AB}\,.\,\frac{\mathrm{ON}}{2}$$

$$= 2\pi r \times \frac{r}{2}$$

$$= \pi r^2.$$

EXERCISES.

Areas of Circles.

(*Take $\pi = \frac{22}{7}$, unless something to the contrary is implied.*)

1. Find the area of a circle whose radius is 3 ft. 6 in. XLI. 1.

2. Taking $\pi = 3{\cdot}1416$, estimate, correct to a sq. in., the area of a circle whose radius is 10 ft. 6 in. XLI. 2.

3. The area of a circle is 3850 sq. links: find its circumference. XLI. 3.

4. Find approximately the area of a running track, 7 ft. wide, surrounding a grass plot 294 ft. in radius. XLI. 4.

5. Find the area of the greatest circle which can be cut out of a triangular piece of paper whose sides are 3, 4, 5 ft. respectively. XLI. 5.

6. Find approximately the radius of a circle whose area is 260·26 sq. yd. XLI. 6.

7. Three circles, each of radius 1 ft., touch each other externally. Find the area of the curvilinear figure included between them. ($\pi = 3{\cdot}1416$.) XLI. 7.

8. The circumference of a circle is 1 ft. Find its area to the nearest hundredth of a sq. in. ($\pi = 3{\cdot}1416$.) XLI. 8.

9. Find, correct to two decimal places, the area included between a circle whose radius is 10 in. and its inscribed square. ($\pi = 3{\cdot}1416$.) XLI. 9.

10. An equilateral triangle and a regular hexagon have the same perimeter. Show that the areas of their inscribed circles are as 4 to 9. XLI. 10.

11. Show how to find the radii of the n concentric circles which divide the area of a given circle into $n+1$ equal parts. XLI. 11.

12. If the radius of a given circle is 1 ft., find to the nearest hundredth of an inch the radii of the two concentric circles which divide its area into 3 equal parts. ($\pi = 3 \cdot 1416$.) XLI. 12.

MISCELLANEOUS EXERCISES.

1. BEAC is a semi-circle whose diameter is BC; D is any point on BC; AD is perpendicular to BDC; EB is equal to AD; and F is on DA produced so that DF is equal to AB; show that CE is equal to CF. XLII. 1.

2. ABC is a triangle and D a point in BC such that AD bisects the angle A. If O be the centre of a circle which touches AB at A and also passes through D, prove that OD and AC are at right angles and find the magnitude of the angle AOD. XLII. 2.

3. The straight line AB of given length moves so that its extremities are respectively upon the two fixed straight lines OC and OD meeting at O. Prove that the centre of the circle circumscribing the triangle OAB lies upon the circumference of a circle whose centre is O. XLII. 3.

4. A moveable circle with constant radius cuts the two fixed straight lines APR, AQS in P, Q, R, and S; prove that for all positions of the circle the sum of the arcs RPQ and PQS is constant. XLII. 4.

5. ABC is an isosceles triangle in which AB is equal to AC, and AD is drawn to meet the base BC in D; show that the centre of the circle described round ABD is at the same distance from AB that the centre of the circle round ADC is from AC. XLII. 5.

6. A quadrilateral ABCD is circumscribed round a circle, touching the circle in E, F, G, H. Find the difference of two opposite angles of ABCD in terms of the difference of two adjacent angles of EFGH. XLII. 6.

7. If a circle roll within another of twice its radius, any point in its circumference will trace out a diameter of the larger circle. XLII. 7.

8. Given a circle and a straight line, find a point in the straight line such that, if tangents be drawn from it to the circle, the chord of contact will subtend a given angle at the circumference. Is it always possible to solve this problem? Give your reasons. XLII. 8.

9. If any two circles be drawn each touching the three sides of a triangle, prove that the circle described on the line joining their centres as diameter passes through two vertices of the triangle. XLII. 9.

10. Two points are given; one is to be the centre of a circle, and the tangent drawn to this circle from the other is to be of given length less than the distance between the given points; show how to draw the circle and the tangent. XLII. 10.

11. The perpendiculars drawn from the angular points A, C of the triangle ABC, to the opposite sides, cut these sides in D and F, and intersect one another in P. Prove that the tangents, at D and F, to the circle which passes through BDPF, pass through the middle point of the side CA; and that the tangents to this circle at P and B are parallel to CA. XLII. 11.

12. A and B are fixed points on a circle whose centre is C, and P is a moving point on the circle; if AP revolves round A at a given rate, find the relative rates at which BP and CP revolve respectively round B and C. XLII. 12.

13. Two circles being given in position and magnitude; draw a straight line cutting them so that the chords in each circle may be equal to a given line, not greater than the diameter of the smaller circle. XLII. 13.

14. The circle inscribed in the triangle ABC touches the side BC at D. Show that the circles inscribed in the triangles BAD and CAD touch each other. XLII. 14.

15. P, C and L denote respectively a given point, circle and unlimited straight line, P and C lying on the same side of L. Find, with proof, a point Q in L such that PQ and a tangent from Q to C (not in the same straight line with PQ) make equal angles with L. Only one solution required. XLII. 15.

16. If O is the centre of the circum-circle of the triangle ABC, and the lines AO, BO, CO produced meet the circumference in A′, B′, C′, show that the triangle A′B′C′ is in every respect equal to ABC. XLII. 16.

17. ABC is a right-angled triangle, A being the right angle. Prove that the hypotenuse BC is equal to the difference between the radius of the inscribed circle of the triangle and the radius of the circle which touches BC and the other two sides produced. XLII. 17.

18. The radii of two intersecting circles are respectively 15 inches and 13 inches, and the common chord of the circles 24 inches long. What length of the line joining their centres lies within both circles? XLII. 18.

19. Describe two circles to touch two given circles, the point of contact with one of these given circles being given. XLII. 19.

20. Prove that, if ABC be a triangle, I the centre of its inscribed circle, D the middle point of BC, L the point where AI produced meets BC, and P the foot of the perpendicular from A on BC, then L lies between P and D, the sides AB and AC being unequal. XLII. 20.

21. Draw a circle to pass through two given points, and to touch a given circle. How must the positions of the points be limited? XLII. 21.

22. Circles through two fixed points A, B intersect fixed straight lines, which terminate at A and are equally inclined to AB on opposite sides of it, in the points L, M; prove that the sum of the lines AL, AM is constant. XLII. 22.

23. AB is a diameter of a circle. PQ is a chord not at right angles to AB. AM and BN are drawn perpendicular to PQ. Prove that PM = NQ. XLII. 23.

24. P and Q are points on a diameter of a circle produced, and are at equal distances from the centre. PR and QT are tangents from P and Q, the points R and T lying on opposite sides of PQ. Prove that PRQT is a parallelogram. XLII. 24.

25. AB and CD are two intersecting chords of a circle. If the arcs AD and BC are together equal to the arcs DB and CA, prove that AB and CD are at right angles to one another. XLII. 25.

26. P, Q, R, S are the orthocentres of the triangles BCD, CDA, DAB, ABC, where A, B, C, D lie on one and the same circle: prove that the quadrilaterals ABCD, PQRS are of the same shape and size. XLII. 26.

27. A, B, C, D are four points taken in order on the circumference of a circle; the straight lines AB, DC produced meet in P, and AD, BC in Q: prove that the straight lines bisecting the angles APC, AQC are at right angles. XLII. 27.

28. ABCD is a parallelogram; AE is at right angles to AB, and CE is at right angles to CB: show that ED produced will cut AC at right angles. XLII. 28.

29. A is a fixed point on the circumference of a circle, and B is any other point on the circumference. AB is produced to C so that BC = AB. Prove that the locus of C is a circle, whose diameter is twice that of the given circle. XLII. 29.

30. If P be the orthocentre of the triangle ABC, the circles circumscribing the triangles ABC, APB, BPC, CPA are equal. XLII. 30.

31. Two opposite sides of a quadrilateral are together equal to the other two, and each of the angles is less than two right angles. Show that a circle can be inscribed in the quadrilateral. XLII. 31.

32. The straight line joining the feet of the perpendiculars from any point of a circle upon two fixed diameters is constant in length. XLII. 32.

33. Prove the converse of the above, viz., that if a straight line of given length slide with its ends on two given intersecting straight lines, the two perpendiculars at its ends to the given lines meet on a fixed circle. XLII. 33.

BOOK IV

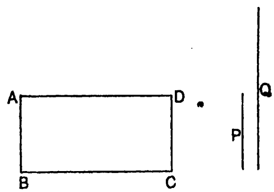
ON RECTANGLES AND AREAS

DEFINITIONS

1. A **rectangle** has been defined as a parallelogram which has one of its angles a right angle.

It must be remembered that all the angles of a rectangle are right angles.

2. A rectangle is said to be **contained by** any two of its conterminous sides.

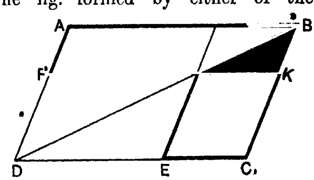


Thus the rect. ABCD is said to be contained by AB and AD; or by AD and DC; and so on.

"*The rect. contained by* AB, BC" is usually abbreviated into "*the rect.* AB . BC."

Also if P and Q are two str. lines respectively equal to AB and BC, the rect. AB . BC is equal to the rect. P . Q.

3. In a parallelogram, the fig. formed by either of the parallelograms about one of its diagonals and the complements is called a **gnomon**.



Thus, in the accompanying diagram, the fig. bounded by the thicker lines is a gnomon, and can be described as the gnomon AKE.

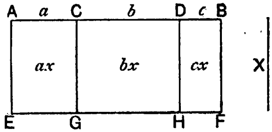
***N.B.*—AB² is often used as an abbreviation for " the square on AB."**

PROPOSITION 1. THEOREM.

If there be two straight lines, one of which is divided into any number of parts, the rectangle contained by the two straight lines is equal to the sum of the rectangles contained by the undivided line and the several parts of the divided line.

Let AB and X be the two str. lines, and let AB be divided into any number of parts AC, CD, DB.

It is reqd. to prove that the rect. X . AB = *rect.* X . AC + *rect.* X . CD + *rect.* X . DB.



From A draw AE at rt. ∠s to AB, and equal to X.

Through E draw EF || to AB;

and through C, D, B draw CG, DH, BF || to AE.

Let AC = a, CD = b, DB = c, and X = x units of length.

The result may be read off immediately from the figure, or algebraically as follows:

rect. AB . X = $(a+b+c)x = ax + bx + cx$

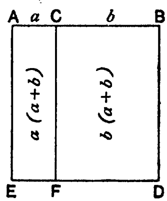
= rect. AC . X + rect. CD . X + rect. DB . X.

Q.E.D.

PROPOSITION 2. THEOREM.

If a straight line be divided into any two parts, the square on the whole line is equal to the sum of the rectangles contained by the whole line and each of the parts.

Let the str. line AB be divided into any two parts AC, CB.



It is reqd. to prove that the sq. on AB = *rect.* AB . AC + *rect.* AB . CB.

On AB describe the sq. ABDE,

and through C draw CF || to AE, or BD.

Let AC = a, CB = b units of length.

The result can be read off from the figure, or deduced algebraically as follows:

sq. on AB = $(a+b)^2 = (a+b)(a+b)$

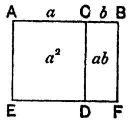
$= (a+b)a + (a+b)b$

= rect. AB . AC + rect. AB . CB. Q.E.D.

PROPOSITION 3. THEOREM.

If a straight line be divided into any two parts, the rectangle contained by the whole line and one of the parts is equal to the square on that part together with the rectangle contained by the two parts.

Let the str. line AB be divided into any two parts AC, CB.



It is reqd. to prove that the rect. AB . AC $=$ *the sq. on* AC $+$ *the rect.* AC . CB.

On AC describe the sq. ACDE, and through B draw BF ‖ to AE, or CD, to meet ED produced at F.

Let AC $= a$ and CB $= b$ units of length.

The result can be read off immediately from the figure or deduced algebraically as follows:

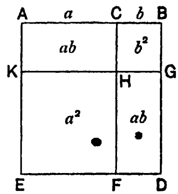
rect. AB . AC $= (a+b)a = a^2 + ab$

$=$ sq. on AC $+$ rect. AC . CB.

PROPOSITION 4. THEOREM.

If a straight line be divided into any two parts, the square on the whole line is equal to the sum of the squares on the two parts together with twice the rectangle contained by the two parts.

Let the str. line AB be divided into any two parts at C.



It is reqd. to prove that the sq. on AB $=$ *the sum of the sqs. on* AC, CB, *together with twice the rect.* AC . CB.

On AB describe the sq. ABDE, and through C draw CF ‖ to AE or BD.

From BD cut off BG equal to BC, and through G draw GHK ‖ to AB or ED, meeting CF at H.

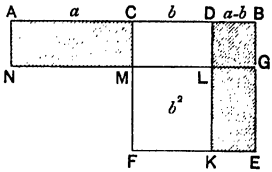
Let AC $= a$ and CB $= b$ units of length.

The result can be read off from the figure, or deduced algebraically as follows:

$$\text{AB}^2 = (a+b)^2$$
$$= a^2 + b^2 + 2ab$$
$$= \text{AC}^2 + \text{BC}^2 + 2 \text{ rect. AC . BC.}$$

PROPOSITION 5. THEOREM.

If a straight line be divided equally and unequally, the rectangle contained by the unequal parts is equal to the difference of the square on half the line and the square on the line between the points of section.



Let the str. line AB be divided equally at C and unequally at D.

It is reqd. to prove that the rect. AD . DB *= the diff. of the sqs. on* CB *and* CD.

On CB describe the sq. CBEF, and through D draw DK ‖ to CF or BE. From BE cut off BG equal to BD, and draw GLMN ‖ to BC or EF, meeting DK at L, CF at M.

Through A draw AN ‖ to CM.

ALGEBRAIC PROOF.

Let AC = CB = a and CD = b units of length.

Then rect. AD . DB $= (a+b)(a-b)$

$= a^2 - b^2$

$= \text{CB}^2 - \text{CD}^2.$

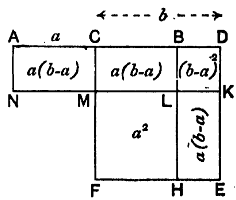
NOTE.—In the figure the shaded rects. AM, DE are equal.

$\therefore$ rect. AD . DB = AL = gnomon CGK

$= \text{CB}^2 - \text{CD}^2.$

PROPOSITION 6. THEOREM.

If a straight line be bisected and produced to any point, the rectangle contained by the whole line thus produced and the part produced is equal to the difference of the square on the line made up of the half and the part produced and the square on half the line bisected.



Let AB be bisected at C and produced to D.

It is reqd. to prove that the rect. AD . DB *= the diff. of the sqs. on* CD *and* CB.

On CD describe the sq. CDEF, and through B draw BH ‖ to DE.

From DE cut off DK equal to DB, and through K draw KLMN ‖ to AD. Draw AN ‖ to CM.

Let $AC = a$, $CD = b$ units of length, so that $DB = b - a$.

From the figure,

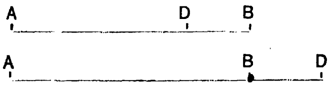
$$\begin{aligned}(b+a)(b-a) &= \text{rect. } AD \,.\, DB \\ &= \text{fig. } AK = \text{fig. } CK + \text{fig. } KH \\ &= \text{fig. } CE - \text{fig. } MH = b^2 - a^2.\end{aligned}$$

ALGEBRAIC PROOF.

$$\begin{aligned}\text{The rect. } AD \,.\, DB &= (b+a)(b-a) \\ &= b^2 - a^2 \\ &= CD^2 - CB^2.\end{aligned}$$

DEFINITION.—If D be any point in the straight line AB, or in AB produced, the lines AD, DB are called the segments of the line AB.

Thus we see that a straight line may be divided into two segments either **internally** or **externally**.



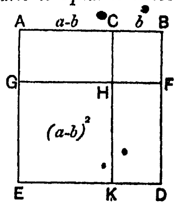
In the figures above, AD and DB are in each case segments of AB.

With this definition, Propositions 5 and 6 may be enunciated as one, as follows:

If a straight line be bisected and divided into two unequal segments (internally or externally), the rectangle contained by the segments is equal to the difference of the squares on half the line and on the line between the points of section.

PROPOSITION 7. THEOREM.

If a straight line be divided into any two parts, the sum of the squares on the whole line and on one of the parts is equal to twice the rectangle contained by the whole and that part, together with the square on the other part.



Let AB be divided into any two parts at C.

It is reqd. to prove that the sqs. on AB *and* BC = *twice the rect.* AB . BC *together with the sq. on* AC.

On AB describe the sq. ABDE, and through C draw CK ∥ to AE or BD.

From BD cut off BF equal to BC, and draw FHG ∥ to AB, cutting CK at H.

ALGEBRAIC PROOF.

Let $AB = a$ and $BC = b$; then

$$\begin{aligned} 2 \text{ rect. } AB \,.\, BC + AC^2 &= 2ab + (a - b)^2 \\ &= 2ab + a^2 - 2ab + b^2 \\ &= a^2 + b^2 \\ &= AB^2 + BC^2. \end{aligned}$$

NOTE.—From the figure, 2 rect. $AB \,.\, BC + AC^2$

$$\begin{aligned} &= 2AF + GK = AF + CD + GK \\ &= AD + CF \\ &= AB^2 + BC^2. \end{aligned}$$

EXERCISES.

1. The square on a straight line is equal to four times the square on half the line. XLV. 1.

2. The square on a straight line is equal to nine times the square on one-third of the line. XLV. 2.

3. Use IV. 5 to make a rectangle equal to the difference of two given squares. XLV. 3.

4. In a right-angled triangle, the square on one of the sides containing the right angle is equal to the rectangle contained by the sum and difference of the hypotenuse and the other side. XLV. 4.

5. Use II. 16 and IV. 4 to prove that, in a right-angled triangle, the square on the perpendicular from the right angle to the hypotenuse is equal to the rectangle contained by the segments of the hypotenuse. XLV. 5.

6. The square on any straight line drawn from the vertex of an isosceles triangle to the base is less than the square on one of the sides by the rectangle contained by the segments of the base. XLV. 6.

7. Prove that, in a right-angled triangle, if a perpendicular be drawn from the right angle to the hypotenuse, the square on one side is equal to the rectangle contained by the hypotenuse and the segment of it adjacent to that side. XLV. 7.

8. If a straight line be divided equally and unequally, the difference between the squares on the unequal parts is equal to twice the rectangle contained by the whole line and the part between the points of section. XLV. 8

9. Show that the sum of the squares on two straight lines is never less than twice the rectangle contained by those straight lines.
XLV. 9.

10. Twice the square on the line joining any point in the hypotenuse of an isosceles right-angled triangle to the right angle is equal to the sum of the squares on the segments of the hypotenuse. XLV. 10.

11. If a straight line be divided into two equal parts and also into two unequal parts, the sum of the squares on the unequal parts is equal to twice the sum of the squares on half the line bisected and on the line between the points of section.

12. If a straight line be bisected and produced to any point, the sum of the squares on the whole line thus produced and on the part produced is equal to twice the sum of the squares on half the line bisected and on the line made up of the half and the part produced.

13. A frame is formed of four rods PQ, QR, RS, SP each of length a, and two rods OQ, OS each of length b (greater than a). The joints O, P, Q, R, S being perfectly free, show that in every position the rectangle OP . OR is equal to $b^2 - a^2$. XLV. 11.

14. Show that of all rectangles having a given area the square is that the sum of whose sides is least. XLV. 12.

15. If a given straight line is divided into any two parts, the sum of the squares on the two parts is a minimum when the parts are equal.
XLV. 13.

16. In AC the diagonal of the square ABCD, a point P is taken. Show that the triangle whose sides are equal to AP, PC and the diagonal of a square described on PB will be right-angled. XLV. 14.

17. ABC is a triangle, and CD is perpendicular to AB ; if the rectangle AD . DB is equal to the square on CD, the angle ACB is a right angle.
XLV. 15.

18. ABCD is a rectangle and AE is drawn to cut CD in E. If F is the middle point of CE, show that the difference between the squares on AC, AE is equal to four times the rectangle CF . DF. XLV. 16.

19. The diagonals of a square ABCD intersect in O, and BD is produced to E so that ED = AD. Show that BE = the diagonal of the square described on OE. XLV. 17.

DEFINITION.—The **orthogonal projection** of a point on a straight line is the foot of the perpendicular drawn from that point to the straight line.

DEFINITION.—The **orthogonal projection of one straight line** upon another is that portion of the second line which is intercepted between the perpendiculars drawn from the ends of the first to the second.

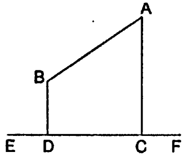


FIG. 1.

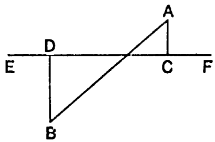


FIG. 2.

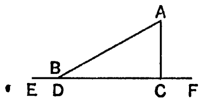
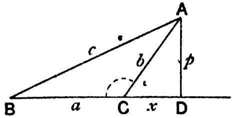


FIG. 3.

Thus in the accompanying diagrams, D is the projection of the pt. B upon the str. line EF: and the projection of the str. line AB upon the str. line EF is, in each case, DC. In diagram 3 the pts. B and D are coincident.

PROPOSITION 8. THEOREM.

In an obtuse angled-triangle, the square on the side subtending the obtuse angle is greater than the sum of the squares on the sides containing the obtuse angle by twice the rectangle contained by either of those sides and the projection on it of the other side.



Let ABC be a △ obtuse-angled at C, and let CD be the projection of AC upon BC.

It is reqd. to prove that AB^2 *is greater than* $BC^2 + CA^2$ *by twice the rect.* BC . CD.

Let $AB = c$, $BC = a$, $CA = b$, $CD = x$, $AD = p$.
Then $AB^2 = BD^2 + DA^2 = (a+x)^2 + p^2 = a^2 + (x^2 + p^2) + 2ax$
$= BC^2 + AC^2 + 2BC \cdot CD$ (for $\angle D$ is a rt. $\angle$).
i.e. AB^2 is greater than $BC^2 + CA^2$ by twice the rect. $CB \cdot CD$.

Q.E.D.

TRIGONOMETRICAL APPLICATION.

In $\triangle ABC$, $\quad \cos C = -\dfrac{CD}{AC}$; $\therefore$ $CD = -AC \cos C$.

$$\therefore\ AB^2 = BC^2 + AC^2 + 2BC \cdot CD$$
$$= BC^2 + AC^2 - 2BC \cdot AC \cos C,$$

or, with the usual notation,

$$c^2 = a^2 + b^2 - 2ab \cos C.$$

This and the following proposition are extensions of the Theorem of Pythagoras.

PROPOSITION 9. THEOREM.

In any triangle, the square on the side opposite an acute angle is equal to the sum of the squares on the other two sides diminished by twice the rectangle contained by either of those sides and the projection on it of the other side.

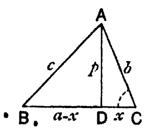


FIG. 1.

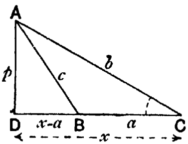


FIG. 2.

Let ABC be any triangle, acute-angled at C, and let CD be the projection of AC upon BC.

It is reqd. to prove that AB^2 *is less than* $BC^2 + CA^2$ *by twice the rect.* $BC \cdot CD$.

Let $AB = c$, $BC = a$, $CA = b$, $AD = p$, $CD = x$.
Then in both figures, $AB^2 = (a \sim x)^2 + p^2$
$= a^2 + (x^2 + p^2) - 2ax = BC^2 + AC^2 - 2BC \cdot CD$

(for $\angle D$ is a rt. $\angle$).

i.e. AB^2 is less than $BC^2 + AC^2$ by twice the rect. $BC \cdot CD$.

Q.E.D.

N.B.— $\sim$ is the symbol denoting "the difference between."

TRIGONOMETRICAL APPLICATION.

In $\triangle ABC$, $\quad \cos C = \dfrac{CD}{CA}$; $\therefore$ $CD = CA \cos C$.

$$\therefore AB^2 = BC^2 + CA^2 - 2BC \cdot CD$$
$$= BC^2 + CA^2 - 2BC \cdot CA \cos C,$$

or, with the usual notation,

$$c^2 = a^2 + b^2 - 2ab \cos C.$$

PROPOSITION 10. (APOLLONIUS' THEOREM.)

In any triangle the sum of the squares on two sides is equal to twice the square on half the base together with twice the square on the median drawn to the base.

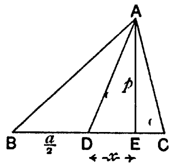


FIG. 1.

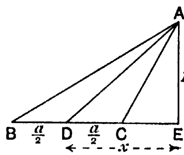


FIG. 2.

Let ABC be any triangle. Draw the median AD to the base BC.

Draw AE perp^r to BC, meeting BC in Fig. 1, and BC produced in Fig. 2 at E.

It is reqd. to prove that $AB^2 + AC^2 = 2 \cdot BD^2 + 2 \cdot DA^2$.

Let $BC = a$, $DE = x$, $AE = p$.

$$AB^2 = BE^2 + AE^2 = \left(\frac{a}{2} + x\right)^2 + p^2$$
$$= \frac{a^2}{4} + x^2 + p^2 + ax = \frac{a^2}{4} + AD^2 + ax.$$

$$AC^2 = CE^2 + AE^2 = \left(\frac{a}{2} \sim x\right)^2 + p^2$$
$$= \frac{a^2}{4} + x^2 + p^2 - ax = \frac{a^2}{4} + AD^2 - ax.$$

Hence, by addition, $AB^2 + AC^2 = \dfrac{a^2}{2} + 2AD^2$

$$= 2 \cdot BD^2 + 2 \cdot AD^2. \qquad \text{Q.E.D.}$$

EXERCISES.

(*N.B.—Algebraic methods will often help to determine a geometrical solution.*)

1. The sides of a triangle are 6, 8, and 12 inches: prove that it is obtuse-angled. XLVI. 1.

2. The sides of a triangle are 9, 12, and 13 inches: prove that it is acute-angled. XLVI. 2.

3. ABCD is a rectangle and P is any point within it: prove that the sum of the squares on PC and PA is equal to the sum of the squares on PB and PD. XLVI. 3.

4. In a triangle ABC the angles B and C are acute: if E and F are the points where the perpendiculars from the opposite angles meet the sides AC, AB, show that the square on BC is equal to the sum of the rectangles AB . BF and AC . CE. XLVI. 4.

5. If the medians of a triangle ABC meet in O, prove that

$$3(OA^2+OB^2+OC^2)=AB^2+BC^2+CA^2. \qquad \text{XLVI. 5.}$$

6. Two sides of a triangle are 8 and 9 inches long, and include an angle of 60 degrees: find, correct to two decimal places, the length of the third side. XLVI. 6.

7. Two sides of a triangle are 6 and 8 inches long, and include an angle of 120 degrees: find, correct to two decimal places, the length of the third side. XLVI. 7.

8. The sides of a triangle are 6, 8, and 10 centimetres long: find, correct to a millimetre, the lengths of its medians. XLVI. 8.

9. The sides of a triangle are 8, 9, and 12 centimetres long: find, correct to a millimetre, the lengths of its medians. XLVI. 9.

10. ABC is a triangle such that AB = 8 cm., BC = 10 cm., and the perpendicular from A on BC divides it in the ratio of 3 to 2: find, correct to two decimal places, the length of AC. XLVI. 10.

11. Given that x, y, z are the lengths of the medians of a triangle, prove that $9 . BC^2 = 8z^2 + 8y^2 - 4x^2$, x being the median drawn to the side BC. XLVI. 11.

12. The medians of a triangle are 5, 7, and 2 inches long; find the length of the side whose median is 2 inches. XLVI. 12.

13. In the triangle ABC, AD and CF are drawn perpendicular to the opposite sides: use IV. 8 and 9 to prove that the rectangle BD . BC is equal to the rectangle BF . BA. XLVI. 13.

14. A point moves so that the sum of the squares of its distances from two fixed points is constant. Prove that the locus of the point is a circle whose centre is at the middle point of the line joining the fixed points. XLVI. 14.

15. The centre of a fixed circle lies at the middle point of the given straight line BC; if A be any point on the circle, prove that the sum of the squares on AB and AC is constant. XLVI. 15.

16. Four times the sum of the squares on the medians is equal to three times the sum of the squares on the sides of a triangle. XLVI. 16.

17. The sum of the squares on the sides of a parallelogram is equal to the sum of the squares on its diagonals. XLVI. 17.

18. The diagonals of the parallelogram ABCD intersect at O, and P is any point on the circumference of a circle whose centre is O; prove that the sum of the squares on PA, PB, PC, PD, is constant. XLVI. 18.

19. The sum of the squares on the two equal sides of an isosceles triangle is less than the sum of the squares on the two sides of any other triangle on the same base and between the same parallels. XLVI. 19.

20. The sum of the squares on the sides of a quadrilateral is equal to the sum of the squares on its diagonals and four times the square on the line joining the middle points of the diagonals. XLVI. 20.

21. The sides of a triangle are 7, 5, and 8 inches: show that one of its angles is two-thirds of a right angle. XLVI. 21.

22. Any point E is taken in the diagonal BD of a square ABCD. Prove that the rectangle BE . ED is equal to the difference of the squares on AB and AE. XLVI. 22.

23. The side BC of a triangle ABC is bisected at D; BD is bisected at H, CD is bisected at K. AH, AK are bisected at L, M. Prove that eight times the difference between the squares on HM, KL is equal to three times the difference between the squares on AB, AC. XLVI. 23.

24. ABC is a triangle having the sides AB and AC equal. On AC a square ADEC is described, and from B a straight line BFG is drawn cutting AC, DE at right angles in the points F and G. Show that the square on BC is equal to twice the rectangle ECFG. XLVI. 24.

25. In any quadrilateral show that the squares on the two sides which subtend an obtuse angle at the intersection of the diagonals are together greater than the squares on the two sides which subtend an acute angle at the same point. XLVI. 25.

26. From the angles of an acute-angled triangle ABC perpendiculars are drawn to the opposite sides, which meet at a point O. Show that the sum of the squares on OA, OB, OC is less than half the sum of the squares on the sides of the triangle. XLVI. 26.

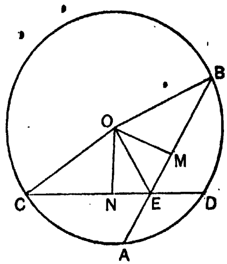
27. ABC is a triangle, and E, F the middle points of the sides AC, AB. Prove that four times the difference between the squares on BE and CF is equal to three times the difference between the squares on BA and CA. XLVI. 27.

28. Construct a triangle having given its area, its base, and the sum of the squares on its sides. XLVI. 28.

PROPOSITION 11. THEOREM.

If two chords of a circle cut one another, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.

Also conversely, if two straight lines cut one another so that the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other, the four extremities of the straight lines are concyclic.



Let the chords AB, CD of the circle ABC cut at E.

It is reqd. to prove that the rect. AE . EB = *the rect.* CE . ED.

Take O the centre of the circle, and draw OM, ON perp^r to AB, CD respectively.

Join OB, OC, OE.

OM is perp^r to AB,

$\therefore$ AB is bisected at M. (III. 1.)

The rect. AE . EB $= (\mathrm{AM} - \mathrm{EM})(\mathrm{BM} + \mathrm{EM})$

$= (\mathrm{BM} - \mathrm{EM})(\mathrm{BM} + \mathrm{EM})$

$= \mathrm{BM}^2 - \mathrm{EM}^2$

$= (\mathrm{OB}^2 - \mathrm{OM}^2) - (\mathrm{OE}^2 - \mathrm{OM}^2)$, for $\angle$ M is a rt. $\angle$,

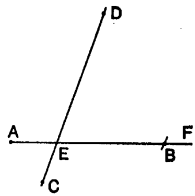
$= \mathrm{OB}^2 - \mathrm{OE}^2$.

In the same way,

the rect. CE . ED $= \mathrm{OC}^2 - \mathrm{OE}^2 = \mathrm{OB}^2 - \mathrm{OE}^2$.

$\therefore$ the rect. AE . EB = the rect. CE . ED. Q.E.D.

Conversely, if the rect. AE . EB = *the rect.* CE . ED, *the pts.* A, B, C, D *will be concyclic.*



If the circle through the pts. A, C, D does not pass through B, let it cut AB again at F.

Then, by the first part of the prop.,

rect. AE . EF = rect. CE . ED

= rect. AE . EB ; *Given.*

∴ EF = EB, which is impossible ;

∴ the circle through A, C, D must also pass through the pt. B, *i.e.* the pts. A, B, C, D are concyclic. Q.E.D.

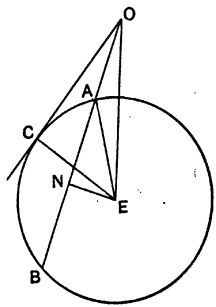
COROLLARY. *The rectangles contained by the segments of all chords of a circle which pass through a given point within it are equal to one another, and equal to the difference of the squares on the radius of the circle and the distance of the given point from the centre.*

This $OE^2 \sim r^2$ is called the **power** of the point E with respect to the circle here. In Prop. 12 it is the power of O.

NOTE.—See also V. 10.

PROPOSITION 12. THEOREM.

If from any point without a circle a secant and a tangent are drawn to the circle, the rectangle contained by the whole secant and its segment external to the circle is equal to the square on the tangent.



Let O be a pt. without the circle ABC, and let OAB be a secant, and OC a tangent to the circle.

It is reqd. to prove that the rect.

OA . OB = *the sq. on* OC.

Take E the centre of the circle, and draw EN perp^r to AB.

Join CE, AE, OE.

EN is perp^r to AB ;

∴ AB is bisected at N ; (III. 3.)

The rect. $OA.OB = (ON - AN)(ON + BN)$
$= (ON - AN)(ON + AN)$
$= ON^2 - AN^2$
$= (OE^2 - EN^2) - (EA^2 - EN^2)$, for $\angle N$ is a rt. $\angle$,
$= OE^2 - EA^2$
$= OE^2 - EC^2$
$= OC^2$, for OC is at rt. $\angle$s to the rad. CE.

Q.E.D.

COROLLARY. *If from any point without a circle any number of secants are drawn to the circle, the rectangles contained by the whole secants and their segments external to the circle are all equal.*

This follows at once from the fact that each rect. is equal to the sq. on the tangent drawn from the given pt.

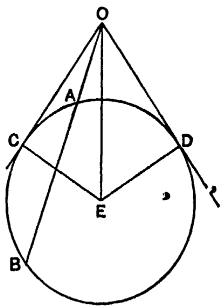
NOTE.—From this corollary and IV. 11 we see that:

If through any given point within or without a circle chords are drawn, the rectangles contained by their segments are equal.

NOTE.—See also V. 11.

PROPOSITION 13. THEOREM.

If from any point without a circle two straight lines are drawn, one of which cuts the circle, and the other meets it, and if the rectangle contained by the whole secant and its segment external to the circle is equal to the square on the line which meets the circle, that line is a tangent.



Let ABC be a circle, and from the point O without it let there be drawn the secant OAB, and the str. line OC to meet the circle at C, so that the rect. $OA \cdot OB = OC^2$.

It is reqd. to prove that OC *is a tangent to the circle.*

Let OD be a tangent to the circle.

Take E the centre, and join EC, EO, ED.

Then $OC^2 =$ the rect. $OA \cdot OB$ *Given.*

$= OD^2$, for OD is a tangent; (IV. 12.)

$\therefore OC = OD$.

Then in the Δs OEC, OED, (1) OC = OD,

(2) EC = ED,

(3) OE is common;

$\therefore$ $\angle$OCE = $\angle$ODE (I. 13.)

= a rt. angle, for OD is a tangent.

$\therefore$ OC is a tangent to the circle ABC. (III. 14.)

Q.E.D.

EXERCISES.

1. AB, CD, two chords of a circle, cut at O so that AO = 3, BO = 5, and CO = 7 inches: find the length of OD. XLVII. 1.

2. AOB is a chord of a circle whose centre is D. AO = 3, BO = 7, and DO = 2 inches: find the length of the radius of the circle. What does your result prove as to the position of D? XLVII. 2.

3. AOB, COD are chords of a circle. AO = 2, BO = 10, and CD = 9 inches. Find the lengths of DO and CO. XLVII. 3.

4. The straight lines OAB, OCD meet a circle at A, B, C, and D. OA = 5, OC = 7, and CD = 3 inches. Find the length of AB. XLVII. 4.

5. OAB is a straight line, such that OA = 4, and AB = 5 inches. Find the length of the tangent drawn from O to a circle through the points A and B. XLVII. 5.

6. The straight line OAB meets the circle whose centre is D at A and B. OA = 4, AB = 5, and OD = 8 centimetres: find, correct to two decimal places, the length of the radius of the circle. XLVII. 6.

7. OAB and OC are two straight lines making an angle of 45 degrees. Draw them making OA = AB = 4 cm. With a centre on the same side of AB as the point C, describe a circle of radius 4 cm. passing through A and B. Measure its intercept on the line OC. XLVII. 7.

8. OAB is a straight line such that OA = 5 and AB = 4 cm. A circle, centre D, is described passing through A and B, OD being 8 cm. long. Find the distance of the point D from the chord AB. XLVII. 8.

9. A brick 4 inches thick is placed to block the wheel of a cart; the face of the brick is 12 inches from the point of contact of the wheel with the ground. Find the radius of the wheel. XLVII. 9.

10. If two circles intersect, their common chord produced bisects their common tangents. XLVII. 10.

11. If two circles intersect, the tangents drawn to the circles from any point in their common chord produced are equal. XLVII. 11.

12. Use IV. 12 to prove that tangents drawn to a circle from an external point are equal. XLVII. 12.

13. ABC is a fixed straight line: find the locus of the points of contact of tangents drawn from C to circles which pass through A and B, which are fixed points. XLVII. 13.

14. From an external point straight lines are drawn to the circumference of a circle: find the locus of their middle points. XLVII. 14.

15. From an external point A tangents AB, AC are drawn to the circle whose centre is O; if OA meets the chord of contact at D, prove that the rectangle OA . OD is equal to the square on the radius. XLVII. 15.

16. Prove IV. 11 when one of the chords is at right angles to the other. XLVII. 16.

17. ABC is a straight line such that AB $=a$ and BC $=b$. On AB as diameter a circle is described, and from C a secant CDE is drawn such that DE $=c$; prove that the length CD may be obtained from the equation $x^2+cx-b(a+b)=0$. XLVII. 17.

18. ABC is a triangle such that AB = BC. A circle, which passes through A and touches BC at its middle point, cuts AB at E. Show that AE is equal to 3EB. XLVII. 18.

19. A straight line AB is divided at any point C; on AC as diameter is described a circle of which any chord AP is drawn. Show that if AP is produced to Q so that the rectangle AP . AQ is equal to the rectangle AB . AC, the point Q lies on a fixed straight line. XVLII. 19.

20. A circle is described so as to pass through the angular points A, B of a square ABCD and the middle point of the side CD. If it cuts DA at O, prove that DA = 4 . DO. XLVII. 20.

21. If two chords of a circle cut one another, the difference of their squares is equal to the difference of the squares of the differences of their segments. XLVII. 21.

22. ABC is a triangle; AP, BQ drawn perpendicular to the opposite sides intersect in O; prove that the rectangle AO . OP equals the rectangle BO . OQ. XLVII. 22.

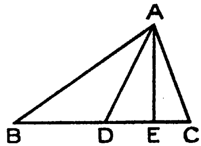
23. Two fixed parallel chords of a fixed circle are cut by a third chord so that the rectangles contained by the segments of the parallel chords are equal. Show that the middle point of the third chord lies upon a fixed straight line. XLVII. 23.

24. ABCD is a straight line divided so that AB = CD, and circles described with centres B and C at the distances BA, CD respectively, intersect in E. Show that AE touches the circle passing through the points B, D, E. XLVII. 24.

25. If three circles cut one another their three common chords are concurrent. XLVII. 25.

Proposition 14. Theorem.

The difference of the squares on two sides of a triangle is equal to twice the rectangle contained by the base and the distance of its middle point from the perpendicular on it from the vertex.



(II. 16.)

Let ABC be a $\triangle$, D the mid. pt. of its base BC, and AE the perp^r from the vertex to the base.

The diff. of the sqs. on AB and AC will be equal to twice the rect. BC . DE.

The $\angle$s at E are rt. $\angle$s;

$$\therefore AB^2 \sim AC^2 = (BE^2 + AE^2) \sim (AE^2 + EC^2)$$
$$= BE^2 \sim EC^2$$
$$= (BE \sim EC)(BE + EC)$$
$$= 2BC \,.\, DE \text{ (for } BE = BD + DE \text{ and } EC = CD \sim DE).$$

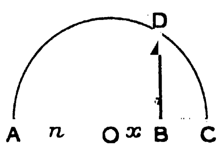
A point moves so that the difference of the squares on its distances from two given fixed points is constant. To prove that the locus of the point is a straight line perpendicular to the line joining the fixed points.

(Use the preceding.)

Note.—$AB^2 \sim AC^2 = AB^2 - AC^2$, if AB is gr. than AC
$= AC^2 - AB^2$, if AC is gr. than AB.

Proposition 15. Problem.

To describe a square equal to a given rectangle.



Draw AB, BC in the same str. line and equal to the sides of the given rect.

On AC as diam^r describe the semicircle ADC.

At B draw BD perp^r to AC to meet the circle at D.

If O is the centre of the circle, join OD.

The sq. on BD *is the sq. reqd.*

Let r = the rad. of the circle, and OB $= x$.

$$\text{Rect. } AB \cdot BC = (r+x)(r-x)$$
$$= r^2 - x^2$$
$$= OD^2 - OB^2$$
$$= BD^2, \text{ for } \angle OBD \text{ is a rt. } \angle;$$

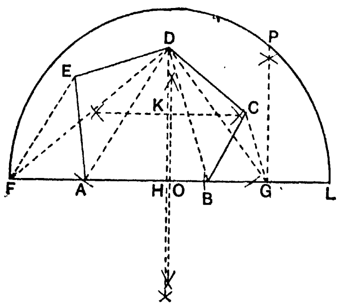
$\therefore$ the sq. on BD is the sq. reqd.

IMPORTANT.—We have here proved that:

If from any point on a circle a perpendicular be drawn to a diameter, the square on the perpendicular is equal to the rectangle contained by the segments of the diameter.

Proposition 16. Problem.

To describe a square equal in area to a given rectilineal figure.



Let ABCDE be the given rectil. fig.

It is reqd. to describe a sq. equal in area to the fig. ABCDE.

Reduce the fig. ABCDE to an equal $\triangle$ DFG. (II. 21.)

Draw DH perp^r to the base FG, and bisect it at K.

In FG produced, make GL equal to HK, and bisect FL at O.

With centre O and rad. OF, or OL, describe a semi-circle FPL, and draw GP $\parallel$ to DH (*i.e.* perp^r to FL) to meet the circle at P.

The sq. on PG is equal in area to the fig. ABCDE.

Join OP.

$$PG^2 = OP^2 - OG^2, \text{ for PGO is a rt. } \angle,$$
$$= OL^2 - OG^2$$
$$= (OL + OG)(OL - OG)$$
$$= (OF + OG)GL, \text{ for FL is bisected at O},$$
$$= \text{rect. } FG \cdot GL,$$
$$= \tfrac{1}{2} \text{ rect. } FG \cdot DH, \text{ for } GL = \tfrac{1}{2}DH,$$
$$= \triangle DFG$$
$$= \text{fig. } ABCDE.$$

∴ the sq. on PG is the sq. reqd.

[*N.B.*—All the lines of construction are shown in the diagram.]

Proposition 17. Problem.

To divide a given straight line into two parts, so that the rectangle contained by the whole and one part may be equal to the square on the other part.

A straight line divided in this manner is said to be divided in **Medial Section.**

[The learner will do well to study the following analysis of this problem. It will explain the construction, and help him to remember it.]

Analysis. If AB is a str. line, we have to divide it at E, so that

$$BA \cdot AE = BE^2. \quad\text{.................................(1)}$$

Let $AB = a$ and $EB = x$, so that $AE = a - x$.

Then, from (1), $\qquad a(a - x) = x^2,$

i.e. $\qquad x^2 + ax = a^2.$

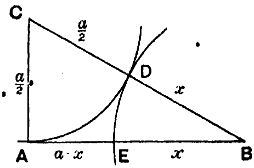
Solving this quadratic equation, we have $x = \dfrac{\pm\sqrt{5} - 1}{2}a.$

Neglect the negative sign, which would give a negative value of x.

Then $$x = \frac{\sqrt{5} - 1}{2}a = \frac{\sqrt{5}a}{2} - \frac{a}{2}.$$

This shows that we have to find a line of length $\frac{\sqrt{5}a}{2}-\frac{a}{2}$, and hence we have the following construction.

Draw AC perp[r] to AB and equal to $\frac{AB}{2}$, *i.e.* $\frac{a}{2}$. Join BC.



$BC^2 = AB^2 + AC^2 = a^2 + \frac{a^2}{4} = \frac{5a^2}{4}$;

$\therefore\ BC = \frac{\sqrt{5}a}{2}$.

To obtain $\frac{\sqrt{5}a}{2}-\frac{a}{2}$, we cut from CB a part CD equal to $\frac{a}{2}$.

Then $BD = BC - \frac{a}{2} = \frac{\sqrt{5}a}{2} - \frac{a}{2}$.

From BA cut off BE equal to BD.

E is the point of division, for $BE = \frac{\sqrt{5}a}{2} - \frac{a}{2}$.

The following may be used instead of the preceding analysis, etc., if thought advisable.

Construction and Proof of Proposition 17.

If AB is the given line, draw AC perp[r] to AB and equal to $\frac{AB}{2}$.

Join CB, and from it cut off CD equal to CA,
and from BA ,, BE equal to BD.

Then the rect. BA . AE *will be equal to* BE^2.

Proof. Let $AB = a$ and $EB = x$.

By Pythagoras' Theorem, $BC^2 = AB^2 + AC^2$, *i.e.* $\left(x+\frac{a}{2}\right)^2 = a^2 + \frac{a^2}{4}$;

$$\therefore\ x^2 + ax + \frac{a^2}{4} = a^2 + \frac{a^2}{4};$$

$$\therefore\ x^2 + ax = a^2;$$

$$\therefore\ x^2 = a^2 - ax = a(a-x); \quad i.e.\ BE^2 = BA \,.\, AE.$$

$\therefore$ the str. line AB is divided as reqd. at the pt. E.

Alternative Proof of Proposition 17.

$$AB \,.\, AE + AB \,.\, EB = AB^2 \qquad \text{(IV. 2.)}$$
$$= CB^2 - CA^2$$
$$= CD^2 + DB^2 + 2 \,.\, CD \,.\, DB - CA^2 \qquad \text{(IV. 4.)}$$
$$= CA^2 + BE^2 + AB \,.\, BE - CA^2,$$
for $2CD = AB$ and $DB = BE$,
$$= BE^2 + AB \,.\, BE.$$

$\therefore$ $AB \,.\, AE = BE^2$, and AB is divided as reqd.

EXERCISES.

1. If a straight line is divided as in IV. 17, prove that the squares on the whole line and one part are equal to three times the square on the other part. XLVIII. 1.

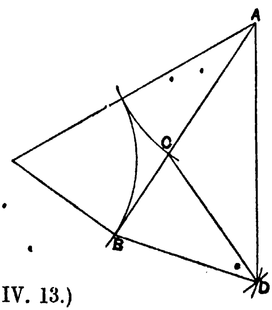
2. If a straight line is divided in medial section, the rectangle contained by the segments is equal to the rectangle contained by the sum and difference of the segments. XLVIII. 5.

3. A straight line 3 inches long is divided internally in medial section; find, correct to two decimal places, the lengths of the segments. XLVIII. 6.

4. Find the lengths of the segments if the line in the above example is divided externally in medial section. XLVIII. 7.

Proposition 18. Problem.

To describe an isosceles triangle having each of the angles at the base double of the third angle.



Take any str. line AB and divide it at C so that the rect. $AB \,.\, BC = AC^2$ (Medial Section).

With centres C and B, describe arcs of radius equal to CA, cutting at D.

Then ABD *is the* $\triangle$ *reqd.*

[*N.B.*—All the construction lines are shown in the diagram.]

$BD^2 = BC \,.\, BA$, by construction.

$\therefore$ BD is a tangent to the circum-circle of the $\triangle ACD$. (IV. 13.)

$\therefore$ the $\angle BDC$ = the $\angle CAD$ (in the alternate segment)

= the $\angle CDA$ (for $CD = CA$);

$\therefore$ the $\angle BDA$ = twice the $\angle CAD$.

Also the $\angle$ABD = the $\angle$BCD (for DB = DC)
= the $\angle$CAD + the $\angle$CDA
= twice the $\angle$CAD.
$\therefore$ ABD is the $\triangle$ reqd.

NOTE.—If n is the number of degrees in the $\angle$A, $5n$ is the number of degrees in the three $\angle$s of the $\triangle$.

$$\therefore 5n = 180;$$

$$\therefore n = 36^\circ, \text{ and each of the angles at the base is } 72^\circ.$$

Thus we see that this proposition enables us to draw geometrically angles of 36° and 72°.

Also notice that $BD = AC = \frac{\sqrt{5}-1}{2} AB$; (IV. 17.)

$$\therefore \frac{BD}{AB} = \frac{\sqrt{5}-1}{2}.$$

EXERCISES.

1. The bisectors of the angles of a regular polygon are concurrent. L. 1.

(If A, B, C be three consecutive angular pts., bisect the $\angle$s A and B by AO and BO. Join CO. Then from the $\triangle$s OAB, OCB it is easily proved that $\angle$BCO = $\angle$BAO = half the $\angle$ of the regular polygon.)

2. The bisectors of the angles of a regular polygon are all equal. L. 2.

3. If a polygon inscribed in a circle is equilateral, it is also equiangular. L. 3.

(Let AB, BC, CD be three consecutive sides of the polygon. The minor arcs AB, CD are equal ; $\therefore$ arc ABC = arc BCD ; $\therefore$ $\angle$ABC = $\angle$BCD.)

4. To inscribe a circle in a given regular polygon.

The bisectors of the angles all meet in one point (Ex. 1).
That one point is equally distant from all the sides. (I. 10.)
$\therefore$ the in-circle is the one whose centre is that point, and whose radius is the perpr from that point to any of the sides.

5. To describe a circle about a given regular polygon.

It has been proved that the bisectors of the $\angle$s of a regular polygon meet in one point.
Let O be that point, and let A, B, C, D, etc., be the vertices.
$\angle$OAB = half the angle of the polygon = $\angle$OBA.

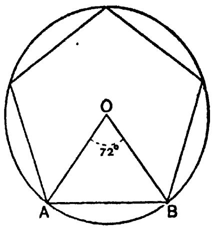
$\therefore$ OA = OB.

Similarly OB = OC = OD, etc.

$\therefore$ the circle described with centre O and radius OA is the circum-circle required.

REGULAR POLYGONS.

If a regular pentagon be inscribed in a circle, each of its sides subtends an angle at the centre of the circle equal to one-fifth of 4 rt. ∠s, *i.e.* 72°.



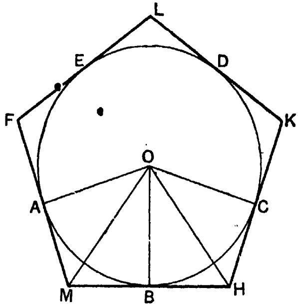
6. To describe a regular pentagon in a given circle.

Let O be the centre of the circle. Draw radii OA, OB including an angle of 72°.
(IV. 18.)

Join AB. Draw chords BC, CD, etc. each equal to AB.

The pentagon so formed can be proved to be regular.

7. To describe a regular pentagon about a given circle.



Let ABCDE be the given circle, O its centre.

Make the ∠s AOB, BOC, etc., each equal to 72°, and so get five equal arcs AB, BC, CD, DE, EA.

Draw tangents FAM, MBH, etc.

In the Δs OBM, OBH,

(1) the ∠MOB = the ∠HOB (halves of equals),
(2) the ∠OBM = ∠OBH (rt. ∠s),
(3) OB is common.

∴ MB = BH.

Similarly each side of the polygon is bisected at its point of contact.

Again MA = MB (tangents);

∴ MF = MH, and similarly for the other sides;

∴ the figure is equilateral.

Moreover its angles must be equal because they are the supplements of the equal angles at the centre.

The same methods may be applied in the case of a regular polygon of n sides by using $360°/n$ instead of $72°$.

8. Inscribe a circle in a given regular pentagon. L. 9.

9. Circumscribe a circle about a given regular pentagon. L. 10.

10. Inscribe a regular quindecagon in a given circle. L. 12.

11. Describe a circle to pass through two given points and to touch a given straight line. Show that there are generally two solutions. L. 13.

(*Analysis.* Let A and B be the given pts. and CD the given line; also let AB produced meet CD at E. Then if the reqd. circle touches CD at F, the rect. EB . EA $=$ EF^2.)

12. Describe a circle to pass through a given point and to touch two given straight lines. L. 14.

(*Analysis.* Let AB, AC be the given lines and D the given pt. The centres of all circles which touch AB and AC lie in AE the bisector of the angle BAC. Draw DF perp^r to AE and produce it, making FG equal to DF. Then, if O is the reqd. centre, OD $=$ OG, *i.e.* the circle passes through G. The problem is now the same as the preceding.)

13. Describe a circle to pass through two given points and to touch a given circle. L. 15.

(*Analysis.* If the circle through the given pts. A and B touches the given circle at P, and the common tangent at P meets AB produced at O, OP^2 $=$ the rect. OA . OB. Hence draw any circle through A and B, cutting the given circle at C and D. Let CD and AB meet at O. From O draw a tangent OP to the given circle. The circle through A, B, and P will be the reqd. circle.)

14. Describe a circle to pass through a given point, to touch a given straight line, and to have its centre on another given straight line. L. 16.

15. O is a given point in a given straight line; with a given centre P describe a circle, cutting the given straight line at A and B, so that the rectangle OA . OB may be equal to a given square. L. 17.

(*Analysis.* If PN be perp^r to OAB, OA . OB $=$ ON^2 $-$ AN^2
$=$ OP^2 $-$ AP^2.)

16. Describe a circle to touch two given straight lines and a given circle. L. 18.

(Let AB, AC be the given str. lines, and D the centre of the given circle. Take D within the $\angle$BAC, and suppose O the centre of the reqd. circle touching AB and AC at B and C. Produce OB to E, and OC to F, making BE $=$ CF $=$ the rad. of the given circle. We then see that OE $=$ OF $=$ OD; $\therefore$ O is the centre of a circle which passes through D and touches str. lines through E and F parallel to the given lines and at distances from them equal to the rad. of the given circle. The problem is now the same as No. 12 above.)

17. Use IV. 12 to make a square equal to a given rectangle. L. 19.

18. Use IV. 12 to describe on a given straight line a rectangle equal to a given square. L. 20.

19. Describe a circle passing through two given points and bisecting the circumference of a given circle. L. 21.

20. Two circles cut a third circle but do not cut one another: find a point such that the tangents from it to all three circles are equal. L. 22.

21. Find a point D in the base BC of a triangle obtuse-angled at A such that the square on AD is equal to the rectangle BD . DC. L. 23.

22. A flagstaff of given height stands on a tower whose height is also given. Find the distance from the foot of the tower of the point at which the flagstaff subtends the greatest angle. L. 24.

23. Find a point in a given straight line produced so that the rectangle contained by the whole and the part produced is equal to a given square. L. 25.

24. Find a point in a given straight line so that the rectangle contained by the two parts is equal to a given square. When is the problem impossible? L. 26.

25. In a circle of radius 7·5 cm. inscribe a triangle equiangular to a triangle whose sides are 3, 4, and 5 cm. long. Measure off and write down the lengths of the sides of the inscribed triangle. L. 27.

26. In a given straight line AB find a point C such that the difference between the squares on AB and AC may be equal to a given square. L. 28.

27. ABC is a triangle inscribed in a circle: in the same circle inscribe a triangle whose sides are at right angles to the sides of the given triangle. L. 29.

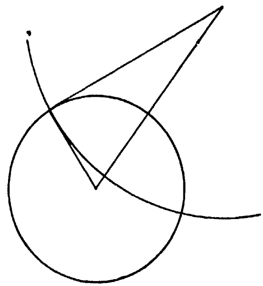
28. In a given straight line AB a point C is taken. Find a second point D in AB such that the sum of the squares on AD, DB may be equal to the square on AC. Within what limits must C lie in order that the problem may be possible? L. 30.

29. Divide a given straight line into two parts such that the rectangle contained by them shall be the greatest possible. L. 31.

30. If the lengths of the sides of a triangle are 10, 24, and 26 feet, find the length of the perpendicular on the side of 26 feet from the opposite angle. L. 32.

ORTHOGONAL CIRCLES.

DEFINITION.—When two circles intersect at a point so that the tangents at that point are at right angles, the circles are said to cut one another **orthogonally.**



Such circles are often called **orthogonal circles.**

We see at once from the diagram that if two circles cut one another orthogonally, the sum of the squares on their radii is equal to the square on the line joining their centres; and, conversely, if the sum of the squares of the radii of two circles is equal to the square on the line joining their centres, the circles cut one another orthogonally.

EXERCISES.

1. If two circles cut one another orthogonally, the tangent to either circle at a point of intersection passes through the centre of the other circle. LI. 1.

2. What is the locus of the centres of circles which cut a given circle orthogonally at a given point? LI. 2.

3. Describe a circle passing through a given point and cutting a given circle orthogonally at a given point. LI. 3.

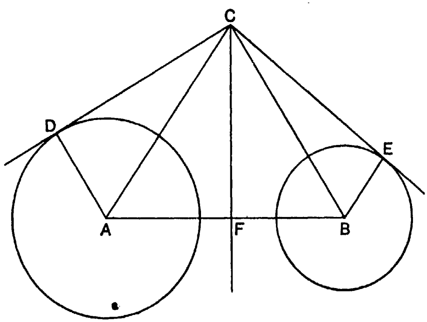
4. In a right-angled triangle ABC, AD is drawn perpendicular to the hypotenuse BC; prove that the circum-circles of the triangles ABD, ACD cut orthogonally. LI. 4.

5. From a point P, without a circle, PA is drawn to touch it in A, and PBC to cut it in B and C. Prove that any circle through B and C cuts at right angles the circle whose centre is P, and radius PA. LI. 5.

6. If C be the centre of a circle, and T any external point, and if the line joining the points of contact of the tangents drawn from T meet CT in U; prove that any circle which passes through T and U cuts the first circle at right angles. LI. 6.

RADICAL AXIS OF TWO CIRCLES.

The locus of a point which moves so that the tangents drawn from it to two given fixed circles are equal, is a straight line perpendicular to the line of centres of the circles.



Let the tangents CD, CE, drawn to the circles whose centres are A and B, be equal. Join AD, AC, AB, BC, BE.

Draw CF at rt. $\angle$s to AB.

$$CD^2 = CE^2;$$

$\therefore$ since the angles at D and E are right $\angle$s,

$$CA^2 - AD^2 = CB^2 - BE^2; \qquad \text{(II. 16.)}$$

i.e. $CA^2 - CB^2 = AD^2 - BE^2$.

But $AD^2 - BE^2$ is constant;

$\therefore$ $CA^2 - CB^2$ is constant.

But since the $\angle$s at F are rt. $\angle$s,

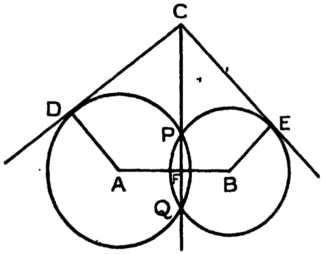
$$CA^2 - CB^2 = (AF^2 + CF^2) - (BF^2 + CF^2)$$
$$= AF^2 - BF^2$$
$$= (AF + FB)(AF - FB);$$

i.e. $(AF + FB)(AF - FB)$ **is constant.**

But $AF+FB$ is constant; $\therefore$ $AF-FB$ is constant;

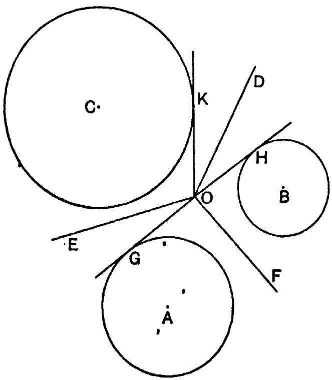
$\therefore$ F is a fixed point;

$\therefore$ the locus of C is a str. line perp^r to AB. Q.E.D.



If the circles cut one another, the locus is the common secant PQ, for $CD^2 = CP \cdot CQ = CE^2$. (IV. 12.)

Definition.—The locus of a point such that the tangents drawn from it to two fixed circles are equal is called the **radical axis** of the circles.



The radical axes of three circles taken in pairs are concurrent.

Let EO be the radical axis of the circles C and A, and DO that of the circles B and C.

The pt. O will be found to be without or within all the circles.

(1) When O is without all the circles.

Draw OG, OH, OK tangents to the circles A, B, C respectively. Since O is a pt. on the radical axis of the circles C and A,

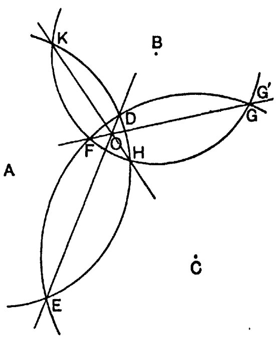
$$OG = OK.$$

In the same way, $OH = OK$;

$$\therefore \; OG = OH;$$

$\therefore$ O is a pt. on the radical axis of the circles A and B; *i.e.* the three radical axes of the circles are concurrent.

(2) In the case when the pt. O lies within all the circles, with the accompanying diagram, let HOK be the common chd. of the circles A and B, DOE the common chd. of the circles A and C.



Let the circles B and C meet at F, and let FO produced meet the circle B in G, and the circle C in G'.

DE, HK are chds. of the circle A; $\therefore$ OD . OE = OH . OK. (IV. 11.)

HK, FG are chds. of the circle B; $\therefore$ OH . OK = OF . OG.

DE, FG' are chds. of the circle C; $\therefore$ OD . OE = OF . OG';

$$\therefore \; OF \,.\, OG = OF \,.\, OG';$$

$$\therefore \; OG = OG';$$

i.e. G and G' coincide at the other common pt. of the circles B and C;

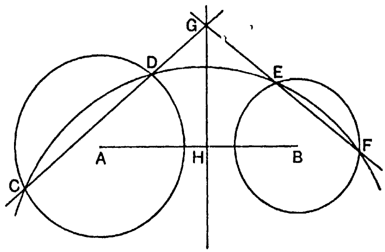
$\therefore$ the radical axes of the three circles are concurrent.

Q.E.D.

DEFINITION.—The point of intersection of the radical axes of three circles is called their **radical centre.**

To draw the radical axis of two given circles.

(1) When the circles cut one another, the radical axis is their common chord.



(2) When the circles do not intersect, describe a third circle cutting the given circles in C, D, E, F.

Let CD and FE meet at G.

The str. line through G perp^r to the line of centres AB will be the radical axis.

The sq. on the tangent from G to the circle A
$=$ GD . GC $=$ GE . GF $=$ the sq. on the tangent from G to the circle B;

$\therefore$ G is a pt. on the radical axis;

$\therefore$ GH drawn perp^r to the line of centres AB is the radical axis reqd.

EXERCISES.

ON THE RADICAL AXIS OF TWO CIRCLES.

1. The radical axis of two circles bisects their common tangents. LII. 1.

2. If tangents are drawn to two circles from any point in their radical axis, and a circle is described with this point as centre and either of the tangents as radius, it will cut the given circles orthogonally. LII. 2.

3. Describe a circle to cut three given circles orthogonally. LII. 3.

(Use the preceding exercises.)

4. The common tangents to three circles which touch one another, two and two, meet at a point. LII. 4.

5. Describe a circle cutting a given circle orthogonally and passing through two given points. LII. 5.

6. Describe a circle cutting two given circles orthogonally and passing through a given point. LII. 6.

BOOK V

ON RATIO AND PROPORTION

BEFORE reading Book V., the student must be acquainted with the algebraical treatment of ratio and proportion, to be found in any Elementary Algebra.

In particular he must be familiar with the following:

1. **We can only find the ratio of two quantities when they are of the same kind.**

For example, we cannot compare in magnitude feet with gallons, or pence with pints. Nor can we compare feet with square feet.

2. *In the ratio* $a : b$, *a and b are called the* **terms** of the ratio, *a being* the **antecedent,** *and b* the **consequent.**

3. *Four magnitudes are said to be in* **proportion** *when the ratio of the first to the second is equal to the ratio of the third to the fourth.*

Thus if $a : b = c : d$, *i.e.* if $\frac{a}{b}=\frac{c}{d}$, a, b, c, d are in proportion.

Here d is said to be a fourth proportional to a, b, and c.

4. *If a, b, c, d are in proportion, $ad=bc$, or the product of the* **extremes** *is equal to the product of the* **means.**

5. *Magnitudes of the same kind are said to be in* **continued proportion** *when the ratio of the first to the second, of the second to the third, of the third to the fourth, and so on, are all equal.*

Thus if a, b, c, d, e, f are in continued proportion,

$$\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=\frac{d}{e}=\frac{e}{f}.$$

The particular case when there are three magnitudes is important, *i.e.* when $\frac{a}{b}=\frac{b}{c}$.

We have then $ac=b^2$, and b is said to be a *mean* proportional between a and c, and c is a *third* proportional to a and b.

6. *When three magnitudes are proportional, the first is to the third as the square of the first is to the square of the second, or as the square of the second is to the square of the third.*

Thus if $\frac{a}{b}=\frac{b}{c}$, $\qquad ac=b^2$;

$$\therefore \frac{a}{c}=\frac{a^2}{ac}=\frac{a^2}{b^2}.$$

The ratio of a^2 to b^2 is called the **duplicate ratio** of a to b.

Thus, *when three magnitudes are proportional, the first is said to have to the third the duplicate ratio of the first to the second.*

7. *Two ratios are said to be* **reciprocal** *when the antecedent of the first is the consequent of the second, and the consequent of the first is the antecedent of the second.*

Thus $\frac{a}{b}$ is the reciprocal of $\frac{b}{a}$, or $a:b$ the reciprocal of $b:a$.

8. If four quantities are proportional, the *antecedents* are said to be corresponding, or homologous terms; so also are the *consequents.*

Thus if $\frac{a}{b}=\frac{c}{d}$, a and c are corresponding terms; so also are b and d.

9. *If four magnitudes are proportional,*

(1) *they are also proportional when taken* **alternately. (Alternando.)**

Thus if $\frac{a}{b}=\frac{c}{d}$, then $\qquad \frac{a}{c}=\frac{b}{d}$.

(2) *They are proportional when taken inversely, i.e.* if $\frac{a}{b}=\frac{c}{d}$, then

$$\frac{b}{a}=\frac{d}{c}.$$

(3) **Addendo.** *With any number of equal ratios of magnitudes of the same kind, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.*

If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, then

$$\frac{a+c}{b+d}=\frac{a}{b}=\frac{c}{d}=\frac{e}{f}, \text{ and } \frac{a+c+e}{b+d+f}=\frac{a}{b}=\frac{c}{d}=\frac{e}{f}.$$

(4) **Componendo.** *If four magnitudes are proportional, the sum of the first and second is to the second as the sum of the third and fourth is to the fourth.*

If $\frac{a}{b}=\frac{c}{d}$, then

$$\frac{a+b}{b}=\frac{c+d}{d} \quad \left[\frac{a}{b}+1=\frac{c}{d}+1;\ \therefore\ \frac{a+b}{b}=\frac{c+d}{d}\right].$$

(5) **Dividendo.** *If four magnitudes are proportional, the difference of the first and second is to the second as the difference of the third and fourth is to the fourth.*

If $\frac{a}{b}=\frac{c}{d}$, then

$$\frac{a-b}{b}=\frac{c-d}{d} \quad \left[\frac{a}{b}-1=\frac{c}{d}-1;\ \therefore\ \frac{a-b}{b}=\frac{c-d}{d}\right].$$

10. *With any number of magnitudes of the same kind, the first is said to have to the last the ratio* **compounded** *of the ratios of the first to the second, of the second to the third, of the third to the fourth, and so on to the last magnitude.*

Thus with three magnitudes a, b, c,

$$\frac{a}{c}=\text{the ratio compounded of } \frac{a}{b} \text{ and } \frac{b}{c}, \quad i.e.\ \frac{a}{c}=\frac{a}{b}\cdot\frac{b}{c}.$$

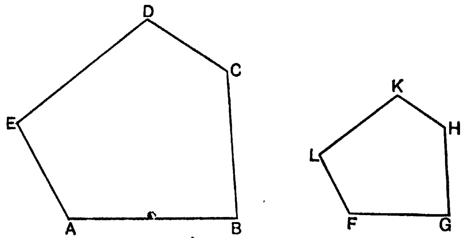
In the same way with four magnitudes a, b, c, d,

$$\frac{a}{d}=\frac{a}{b}\cdot\frac{b}{c}\cdot\frac{c}{d}, \quad \text{and so on.}$$

DEFINITIONS.

1. Two rectilineal figures are said to be equiangular when the angles of the first, *taken in order*, are respectively equal to the angles of the second, taken in order.

2. Rectilineal figures are said to be similar when they are equiangular to one another and have their sides about the equal angles *taken in order* proportionals.



Thus the polygons ABCDE, FGHKL are similar if the ∠s at A, B, C, D, E are respectively equal to the ∠s at F, G, H, K, L, and if also the ratios, $\frac{AB}{FG}$, $\frac{BC}{GH}$, $\frac{CD}{HK}$, $\frac{DE}{KL}$, $\frac{EA}{LF}$ are equal to one another.

3. In similar rectilineal figures, the angles which are equal to one another are said to be **corresponding angles**; and the sides which are adjacent to corresponding angles at both ends are said to be **corresponding, or homologous sides.**

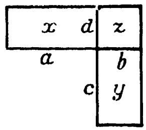
Thus in the above diagram, AE and FL are corresponding or homologous sides; for AE is adj. to the ∠s A and E, and FL is adj. to the corresponding ∠s F and L.

4. If two similar rectilineal figures have their corresponding, or homologous sides parallel and drawn in the same direction, they are said to be similarly situated.

5. Two rectilineal figures which have their sides about an angle in each proportional in such a manner that a side of the first is to a side of the second as the remaining side of the second is to the remaining side of the first, are said to have their sides about those angles **reciprocally proportional.**

Application of paragraph 4, p. 181, to Geometry.

If four straight lines are proportional, the rectangle contained by the extremes is equal to the rectangle contained by the means. Conversely, if the rectangle contained by the extremes is equal to the rectangle contained by the means, the four straight lines are proportional.



If a, b, c, d represent straight lines and $\frac{a}{b}=\frac{c}{d}$,

then the rect. $a \, . \, d =$ the rect. $b \, . \, c$.

Let x be a rect. whose sides are a and d in length,
and y „ „ „ b and c „

Arrange the rects. so that a and b are in the same str. line; c and d will then be in a str. line.

Let z represent the rect. whose sides are b and d.

Then, from the figure,

$$\frac{x}{z}=\frac{ad}{bd}=\frac{a}{b},$$

$$\frac{y}{z}=\frac{cb}{db}=\frac{c}{d}=\frac{a}{b}; \qquad \textit{Given.}$$

$$\therefore \; x=y, \quad i.e. \; ad=bc.$$

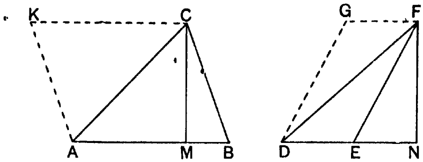
Conversely, if $ad=bc$, $\frac{a}{b}=\frac{c}{d}$.

$$ad=bc; \qquad \textit{Given.}$$

$$\therefore \; \frac{ad}{bd}=\frac{bc}{bd}; \quad \therefore \; \frac{a}{b}=\frac{c}{d}. \qquad \text{Q.E.D.}$$

PROPOSITION 1. THEOREM.

The areas of triangles (or parallelograms) of equal altitudes are to one another as their bases.



(a) Let ABC, DEF be two $\triangle$s of equal altitudes CM, FN.

It is reqd. to prove that $\dfrac{\triangle ABC}{\triangle DEF} = \dfrac{AB}{DE}$.

The area of a $\triangle$ is measured by half the product of the measures of the lengths of its base and altitude (II. 11.).

$$\therefore \frac{\triangle ABC}{\triangle DEF} = \frac{\frac{1}{2}AB \,.\, CM}{\frac{1}{2}DE \,.\, FN} = \frac{AB}{DE}, \text{ for } CM = FN \text{ by hyp.}$$

(b) Let ABCK, DEFG be two parallelograms of equal altitudes CM . FN.

It is reqd. to prove that $\dfrac{\text{par}^{m}\ ABCK}{\text{par}^{m}\ DEFG} = \dfrac{AB}{DE}$.

The area of a par^{m} is measured by the product of the measures of the lengths of its base and altitude.

$$\therefore \frac{\text{par}^{m}\ ABCK}{\text{par}^{m}\ DEFG} = \frac{AB \,.\, CM}{DE \,.\, FN} = \frac{AB}{DE}, \text{ for } CM = FN \text{ by hyp.}$$

Q.E.D.

COROLLARY. Triangles, or parallelograms, on equal bases are to one another as their altitudes.

This is proved in the same way as the proposition.

INCOMMENSURABLE MAGNITUDES.

The student is strongly recommended to defer the study of incommensurable magnitudes until he has thoroughly mastered all the propositions for commensurable magnitudes.

Proofs for Incommensurables will be found at the end of Book V.

The ratio of two quantities cannot always be expressed by the ratio of two whole numbers. Thus the ratio of the diagonal of a square to one of its sides is $\frac{\sqrt{2}}{1}$; and we cannot find any fraction exactly equal to $\sqrt{2}$.

DEFINITION.—Magnitudes whose ratio cannot be exactly expressed by the ratio of two whole numbers are said to be **incommensurable.**

Although we cannot determine the ratio of two incommensurable magnitudes *exactly*, we can do so to any required degree of approximation.

Thus by calculating the value of $\sqrt{2}$ to any required number of decimal places, we can obtain, as nearly as we please, the value of the ratio $\frac{\sqrt{2}}{1}$.

Thus $\qquad \frac{\sqrt{2}}{1}=\frac{1\cdot 41}{1}=\frac{141}{100}$ to two dec. places.

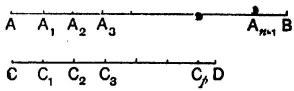
Also $\qquad \frac{\sqrt{2}}{1}=\frac{1\cdot 414}{1}=\frac{1414}{1000}$ to three dec. places,

and $\qquad \frac{\sqrt{2}}{1}=\frac{1\cdot 41421}{1}=\frac{141421}{100000}$ to five dec. places,

and so on.

Thus in a rect. whose adjacent sides are $\sqrt{2}$ and $\sqrt{3}$, $\sqrt{2}\times\sqrt{3}$ $(=\sqrt{6})$ may be taken as a measure of its area. We cannot determine the *exact* value of $\sqrt{6}$, but by calculating $\sqrt{6}$ to any number of dec. places, we can determine its value to any reqd. degree of approximation.

In dealing with incommensurable straight lines the following plan may be adopted.



Let AB, CD be two incommensurable str. lines. Divide AB into any number (n) of equal parts, AA_1, A_1A_2 ... $A_{n-1}B$.

Along CD set off parts CC_1, C_1C_2 ... each equal to AA_1; and let C_p be the pt. of division nearest to D, so that C_pD is less than AA_1. Then

$$\frac{AB}{CC_p}=\frac{n\,.\,AA_1}{p\,.\,CC_1}=\frac{n}{p}.$$

By increasing n indefinitely we can make C_pD as small as we please,

i.e. we can make CC_p (a magnitude commensurable with AB) differ as little as we please from CD (which is incommensurable with AB).

$\therefore$ we can make the ratio $\frac{AB}{CC_p}$ differ as little as we please from the ratio $\frac{AB}{CD}$.

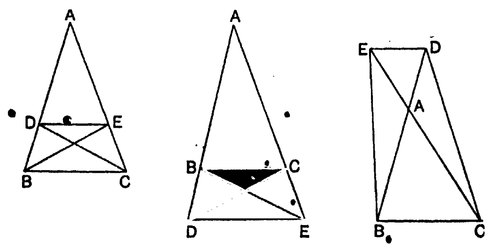
In other words, by making our unit of measurement indefinitely small, we can make the ratio of two incommensurable str. lines differ as little as we please from the ratio of two commensurable str. lines.

Hence, by looking upon the unit of measurement as being indefinitely small, a ratio of two incommensurable str. lines may be treated in the same way as the ratio of two commensurable str. lines.

PROPOSITION 2. THEOREM.

If a straight line is drawn parallel to one side of a triangle, it cuts the other sides, or those sides produced, proportionally.

Conversely, if the sides, or the sides produced, are cut proportionally, the straight line joining the points of section is parallel to the remaining side of the triangle.



(1) Let DE be drawn par[l] to BC, one of the sides of the $\triangle ABC$

It is reqd. to prove that $\frac{AD}{DB} = \frac{AE}{EC}$.

Join BE, DC.

$\frac{AD}{DB}=\frac{\triangle ADE}{\triangle DBE}$, for $\triangle$s ADE, DBE are of the same altitude, (V. 1.)

$=\frac{\triangle ADE}{\triangle DCE}$, for $\triangle$s DBE, DCE are on the same base and between ||s, (II. 11, *Cor.* 2.)

$=\frac{AE}{EC}$, for $\triangle$s ADE, DCE are of the same altitude. (V. 1.)

(2) Let $\frac{AD}{DB}=\frac{AE}{EC}$.

It is reqd. to prove that DE *is* || *to* BC.

$\frac{\triangle DBE}{\triangle ADE}=\frac{DB}{DA}$, for $\triangle$DBE, ADE are of the same altitude, (V. 1.)

$=\frac{CE}{EA}$, *Given.*

$=\frac{\triangle CDE}{\triangle ADE}$, for $\triangle$s CDE, ADE are of the same altitude; (V. 1.)

$\therefore \triangle DBE=\triangle CDE$,

and they are on the same base and on the same side of it;

$\therefore$ DE is || to BC. (II. 13.)

Q.E.D.

COR. 1. *If a str. line parallel to the base* BC *of a* $\triangle$ABC *cut the sides* AB, AC *in* D *and* E,

$$\frac{AB}{AD}=\frac{AC}{AE}.$$

By the above prop., $\frac{AD}{DB}=\frac{AE}{EC}$;

$$\therefore \frac{DB}{AD}=\frac{EC}{AE};$$

$\therefore$ componendo, $\frac{DB+AD}{AD}=\frac{EC+AE}{AE}$,

i.e. $\frac{AB}{AD}=\frac{AC}{AE}$.

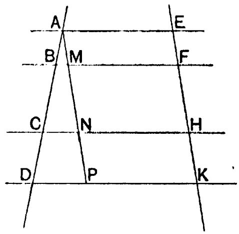
Conversely, *if* D, E *be pts. in the sides* AB, AC *of a* △, *so that* $\frac{AB}{AD}=\frac{AC}{AE}$, *then* DE *is* ∥ *to* BC.

For, dividendo, $$\frac{AB-AD}{AD}=\frac{AC-AE}{AE},$$

i.e. $$\frac{BD}{DA}=\frac{CE}{EA};$$

∴ DE is ∥ to BC.

COR. 2. *Transversals are divided proportionally by parallel straight lines.*



Let ABCD, EFHK be cut by ∥ str. lines as shown in the diagram.

Draw AMNP ∥ to EFHK as shown.

The figs. AF, MH, NK are par^{ms}.

$\therefore \frac{AB}{EF}=\frac{AB}{AM}=\frac{BC}{MN}=\frac{BC}{FH}$, and so on.

EXERCISES.

1. The four triangles into which a quadrilateral is divided by its diagonals are proportional. LIII. 1.

2. The diagonals of a trapezium cut one another proportionally. LIII. 2.

3. From any point in the base of a triangle straight lines are drawn parallel to the sides. Find the locus of the intersection of the diagonals of the parallelogram so formed. LIII. 3.

4. Prove, as in Proposition 2, that the line joining the middle points of two sides of a triangle is parallel to the base and is equal to one half of the base. LIII. 4.

5. Three parallel straight lines intercept on any other two straight lines portions which are proportional to one another. LIII. 5.

6. From a point E in the common base of two triangles ACB, ADB, straight lines are drawn parallel to AC, AD, meeting BC, BD at F and G: prove that FG is parallel to CD. LIII. 6.

7. Through a point P within the angle ABC, draw a straight line APC so that AP = PC. LIII. 7.

8. From a given point D in the side AB of the triangle ABC, draw a straight line to meet AC produced so that it is bisected by BC. (Draw DE parallel to BC to meet AC at E; in AC produced take CF = CE. DF will be the reqd. line.) LIII. 8.

9. A straight line drawn parallel to BC, one of the sides of the triangle ABC, meets AB at D and AC at E; if BE and CD meet at F, then the triangle ADF is equal to the triangle AEF. LIII. 9.

10. If from a point O in the base of a triangle ABC, OM and ON are drawn parallel to the sides AB, AC respectively, then the area of the triangle OMN is a mean proportional between the areas of the triangles BNO and CMO. LIII. 10.

11. In the sides AC, AB of a triangle points D, E are taken such that BE is to EA as AD is to DC as 1 is to 2; compare the areas of the triangles BED, CED. LIII. 11.

12. ABC is a straight line bisected in B. DE, FG are any lines parallel to ABC, FG lying between DE and ABC. DA, DB, EB, EC meet FG in F, H, K, G respectively: and the sum of FH and KG is equal to HK. Prove that AH, CK meet on DE. LIII. 12.

13. Equal lengths AK, CL are marked off on the diagonal of the parallelogram ABCD, and DK, DL, when produced, cut the sides AB, BC in F, G respectively. Show that FG is parallel to AC. LIII. 13.

14. Straight lines AOB, COD intersect at O, and

$$AO : OB :: CO : OD;$$

if P, Q are the middle points of AB, CD, prove that PQ is parallel to AC and BD. LIII. 14.

15. In a parallelogram, either of the complements is a mean proportional between the two parallelograms about the diameter.
LIII. 15.

16. Prove by V. 1, that the medians of a triangle are concurrent.
LIII. 16.

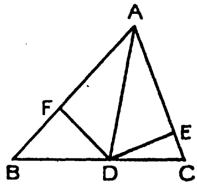
17. ABCD is a parallelogram, and E a point in AD such that AE is to ED as 1 is to 2: prove that the triangle ABE is one-sixth of the whole parallelogram. LIII. 17.

18. E is a point in the diagonal BD of the parallelogram ABCD such that BE is to ED as 4 is to 3: prove that the triangle AED is to the parallelogram as 3 is to 14. LIII. 18.

PROPOSITION 3. THEOREM.

If the vertical angle of a triangle is bisected by a straight line which cuts the base, the segments of the base are to one another in the same ratio as the other sides of the triangle.

Conversely, if the base is divided internally so that its segments are to one another in the same ratio as the adjacent sides of the triangle, the straight line drawn from the vertex to the point of section bisects the vertical angle.



(1) Let AD bisect the vertical $\angle$A of the $\triangle$ABC and cut the base in D.

It is reqd. to prove that

$$\frac{BD}{DC} = \frac{BA}{AC}.$$

Draw DE perp^r to AC and DF perp^r to AB.

DE = DF, for D lies on the bisector of the $\angle$BAC.

$$\therefore \frac{\triangle DAB}{\triangle DAC} = \frac{BA}{AC}, \text{ since the altitudes DF, DE are equal.}$$

But $$\frac{\triangle DAB}{\triangle DAC} = \frac{BD}{DC}; \qquad \text{(V. 1.)}$$

$$\therefore \frac{BA}{AC} = \frac{BD}{DC}.$$

Q.E.D.

(2) Let $\frac{BD}{DC} = \frac{BA}{AC}$.

It is reqd. to prove that AD *bisects the* $\angle$BAC.

$$\frac{BA}{AC} = \frac{BD}{DC} = \frac{\triangle DAB}{\triangle DAC}, \text{ for these } \triangle\text{s have the same alt.,}$$

$$= \frac{\frac{1}{2}DF \cdot AB}{\frac{1}{2}DE \cdot AC};$$

$$\therefore DF = DE,$$

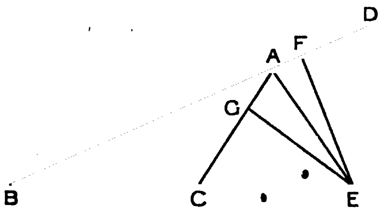
i.e. AD bisects the $\angle$BAC.

Q.E.D.

PROPOSITION 4. THEOREM.

If the exterior vertical angle of a triangle is bisected by a straight line which cuts the base produced, the external segments of the base are to one another in the same ratio as the other sides of the triangle.

Conversely, if the external segments of the base have to one another the same ratio as the other sides of the triangle, the straight line drawn from the vertex to the point of section bisects the exterior vertical angle.



(1) Let the side BA of the $\triangle ABC$ be produced to D, and let AE bisect the ext. $\angle CAD$ and meet the base BC produced at E.

It is reqd. to prove that $\dfrac{BE}{EC}=\dfrac{BA}{AC}$.

Draw EF perp^r to BD and EG perp^r to AC.

EF = EG, for E lies on the bisector of the $\angle CAD$.

$$\therefore \frac{\triangle BAE}{\triangle CAE}=\frac{BA}{AC}, \text{ since the altitudes EF, EG are equal.}$$

But $$\frac{\triangle BAE}{\triangle CAE}=\frac{BE}{CE}; \qquad \text{(V. 1.)}$$

$$\therefore \frac{BE}{CE}=\frac{BA}{AC}.$$

Q.E.D.

(2) Let $\frac{BE}{EC}=\frac{BA}{AC}$.

It is reqd. to prove that AE *bisects the* $\angle$CAD.

$$\frac{BA}{AC}=\frac{BE}{EC}=\frac{\triangle BAE}{\triangle CAE}, \text{ for these } \triangle\text{s have the same alt.,}$$

$$=\frac{\frac{1}{2}BA \,.\, EF}{\frac{1}{2}CA \,.\, EG};$$

$$\therefore\ EF=EG,$$

i.e. AE bisects the $\angle$CAD. Q.E.D.

NOTE.—Propositions 3 and 4 may be included under one enunciation as follows: *If the interior or exterior vertical angle of a triangle is bisected by a straight line which also cuts the base, the base is divided internally or externally into segments which are proportional to the sides; and conversely.*

EXERCISES.

1. If a straight line bisects both the vertical angle and the base of a triangle, the triangle is isosceles. LIV. 1.

2. ABC is a triangle whose base BC is bisected at D; the angles ADB, ADC are bisected by DE, DF meeting at AB, AC at E, F. Prove that EF is parallel to BC. LIV. 2.

3. **Employ Proposition 3 to prove that the bisectors of the angles of a triangle are concurrent.** LIV. 3.

4. If O is the centre of the inscribed circle of the triangle ABC, and AO produced meets BC at D, prove that AO is to OD as the sum of AB and AC is to the base BC. LIV. 4.

5. If the bisectors of the angles A and C of the quadrilateral ABCD meet on BD, prove that the bisectors of the angles B and D meet on AC. LIV. 5.

6. CD is a chord of a circle at right angles to a diameter AB. Through E any point in CD, AE, BE are drawn to meet the circle in F and G. Prove that the quadrilateral CFDG has any two of its adjacent sides in the same ratio as the other two. LIV. 6.

7. One circle touches another internally at O. A straight line touches the inner circle at C, and meets the outer one at A and B: prove that OA is to OB as AC is to CB. LIV. 7.

8. Prove that the radius of the circle inscribed in any isosceles triangle has the same ratio to the perpendicular altitude as the base of the triangle has to its perimeter. LIV. 8.

9. BD is the perpendicular let fall from one end of the base upon the straight line bisecting the vertical angle BAC of a triangle. If AB be three times as long as AC, prove that AD will be bisected at the point where it cuts the base. LIV. 9.

10. A point E is taken in AB the side of a square ABCD such that BE is to EA as 3 is to 4, and F is taken in AD such that AF is to FD as 2 is to 5. Find the ratio in which EF is cut by the diagonal AC. LIV. 10.

11. The angle A of a triangle ABC is bisected by AD, which cuts the base at D, and O is the middle point of BC; show that OD bears the same ratio to OB that the difference of the sides bears to their sum. LIV. 11.

12. The bisector of the vertical angle of a triangle and the bisectors of the exterior angles at the base are concurrent. LV. 1.

13. From P any point on a circle of diameter AB, PC and PD are drawn equally inclined to AP on opposite sides of it and meeting AB at C and D. Prove that AC is to CB as AD is to BD. LV. 2.

(Prove that PA, PB bisect interior and exterior angles of the triangle CPD.)

14. On a given base AB a triangle APB is described whose sides AP, PB are in a given ratio : find the locus of P. LV. 3.

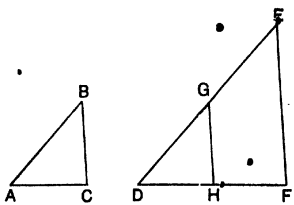
(If the bisectors of the interior and exterior angles at P meet AB at C and D, it will be seen that C and D are fixed points, and that the angle CPD is a right angle. The locus therefore is a circle on CD as diameter.)

15. From the point O straight lines are drawn making the angles AOB, BOC, COD each equal to half a right angle, and they are cut by a straight line ABCD, which makes AOD an isosceles triangle. Prove that AB or CD is a mean proportional between AD and BC. LV. 4.

16. Determine the locus of the vertex of a triangle on a given base, when the straight line drawn from the vertex to a fixed point in the base bisects the vertical angle. LV. 5.

PROPOSITION 5. THEOREM.

If two triangles are equiangular to one another, they are also similar.



Let the $\triangle$s ABC, DEF be equiangular, having $\angle A = \angle D$, $\angle B = \angle E$, and consequently $\angle C = \angle F$.

It is reqd. to prove that

$$\frac{BC}{EF} = \frac{CA}{FD} = \frac{AB}{DE}.$$

Apply the $\triangle ABC$ to the $\triangle DEF$ so that A falls on D, and AC falls along DF, C falling at H.

Then AB falls along DE, for $\angle BAC = \angle EDF$.

Let G be the new position of B, so that GH is the new position of BC.

$$\angle DGH = \angle B = \angle DEF;$$

$$\therefore \text{ GH is parallel to EF}; \qquad \text{(I. 4.)}$$

$$\therefore \frac{DG}{DE} = \frac{DH}{DF}, \qquad \text{(V. 2, } Cor.\text{)}$$

$$i.e. \quad \frac{AB}{DE} = \frac{AC}{DF}.$$

In the same way, by applying the $\triangle ABC$ to the $\triangle DEF$, so that $\angle C$ coincides with $\angle F$, we may prove that

$$\frac{AC}{DF} = \frac{BC}{EF};$$

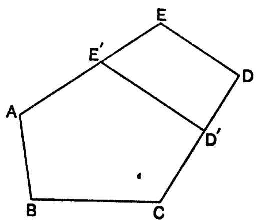
$$\therefore \frac{BC}{EF} = \frac{CA}{FD} = \frac{AB}{DE}.$$

Q.E.D.

COROLLARY. Taking the ratios alternately,

$$\frac{BC}{CA} = \frac{EF}{FD}, \quad \frac{CA}{AB} = \frac{FD}{DE}, \quad \text{and} \quad \frac{AB}{BC} = \frac{DE}{EF}.$$

NOTE.—Equiangular quadrilaterals or polygons are not necessarily similar.



In the accompanying diagram, where D′E′ is parallel to DE, the figs. ABCDE, ABCD′E′ are equiangular, but their sides about all their equal angles are obviously not proportional.

E.g. $\frac{BC}{CD'}$ is not equal to $\frac{BC}{CD}$.

EXERCISES.

1. The straight lines joining the middle points of the sides of a triangle divide the triangle into four triangles, each similar to the whole triangle, and equal in all respects, and therefore each equal to one quarter of the whole triangle. LVI. 1.

2. The straight line OAB cuts a circle at A and B, and OC is a tangent; prove that the rect. OA.OB = the sq. on OC, by making the points C and D coincide where OCD is a secant. LVI. 3.

3. From the top of a mast h feet above sea-level a man can just see a ship at a distance of l miles. If r miles is the radius of the earth, prove that $l=\sqrt{\frac{rh}{2640}}$ approximately. LVI. 4.

4. The Centroid of a triangle is one of the points of trisection of each median. LVI. 5.

Prove this by using similar triangles.

[Def.: The point of intersection of the medians of a triangle is called its **centroid.**]

5. A man 6 feet in height stands at a distance of 4 feet from a lamp-post 12 feet high: find the length of his shadow. LVI. 6

6. A man 6 feet high has a shadow, 10 feet long, cast by a lamp at the top of a post 12 feet in height: how far is the man standing from the post? LVI. 7.

7. The sun-shadow of a pole 10 feet high is 20 feet long, when that of a wall is 60 feet long: find the height of the wall. LVI. 8.

8. Two similar triangles are described on bases of 5 and 3 cm.; if the other sides of the first triangle are 9 and 7 cm. long, find the lengths of the remaining sides of the second triangle. LVI. 9.

9. Two straight lines OAB, OCD cut a circle at A, B, D and C. If OA = 5, OC = 4, CD = 8, and BD = 7 feet, find the lengths of AB and AC. LVI. 10.

10. A rectangle ABCD, whose adjacent sides are 6 and 8 feet long respectively, is folded so that the point A falls on the point C. Find the length of the crease. LVI. 11.

11. ABC is an acute-angled triangle, and AD is drawn perpendicular to BC. The triangle is folded so that the point A falls on the point D. Find the length of the crease. LVI. 12.

12. In the triangle ABC, AB = 13, BC = 5, and CA = 12 inches. The triangle is folded so that the point A falls on the point B: find the length of the crease. LVI. 13.

13. ABC is a triangle whose base BC is 9 in. long; a straight line is drawn parallel to the base cutting AB in the ratio of 4 to 3; find the length intercepted by the sides of the triangle. LVI. 14.

14. If one of the two parallel sides of a trapezium is double the other, the diagonals intersect at a point of trisection. LVI. 15.

15. If three straight lines meet at a point, they intercept on any parallel straight lines portions which are proportional to one another. LVI. 16.

16. A common tangent to two circles cuts their line of centres internally or externally in the ratio of the radii. LVI. 17.

17. If any straight line is drawn parallel to the base of a triangle, the portion intercepted by the sides is bisected by the median to the base. LVI. 18.

18. If D is any point in the base BC of a triangle ABC, the radii of the circles circumscribing the triangles ADB, ABC are proportional to the sides AB, AC. LVI. 19.

19. If the perpendiculars from two fixed points on a straight line passing between them are in a given ratio, the straight line will pass through a third fixed point. LVI. 20.

20. Straight lines AOD, BOE intersect at O so that AO = 2OD and BO = 2OE; AE and BD are drawn and produced to meet at C; prove that AC and BC are bisected at E and D. LVI. 21.

21. AB is parallel to CD and is bisected at O by the line QOPD: CA and CB cut this line in Q, P: prove that QO is to OP as QD is to PD. LVI. 22.

22. Verify the following construction to trisect a given finite straight line AB. Produce BA to C, making AC = AB, and draw any two parallel straight lines BD, CE to meet any other two parallel straight lines AD, BE in D, E respectively: then DE trisects AB. LVI. 23.

23. In a triangle ABC, AD bisects the angle A, and from C a perpendicular is drawn to AD meeting it in N. If O is the middle point of BC, prove that ON is half the difference of the sides AB, AC. LVI. 24.

24. A and B are the points of intersection of two circles; CAD is a straight line meeting the circles again at C and D; and BC, BD are drawn: prove that BC is to BD as the diameter of the circle ABC is to that of the circle ABD. LVI. 25.

25. If D, E, F are the centres of the escribed circles touching the sides BC, CA, AB respectively of a triangle ABC, prove that FA and FB are inversely proportional to FE and FD. LVI. 26.

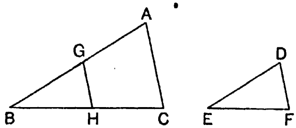
26. The bisector of the angle BAC of a triangle meets BC in D, and meets in E the straight line which bisects BC at right angles; prove that the rectangle ED . EA is equal to the square on EB. LVI. 27.

27. The diagonal BD of a parallelogram ABCD is divided in E so that BE is one-third of ED, and AE, DC produced meet in F. Show that FC is double of AB. LVI. 28.

Proposition 6. Theorem.

(Converse of Prop. 5.)

If two triangles have the sides, taken in order, about each of their angles proportional, the triangles are similar, those angles being equal which are opposite to the corresponding or homologous sides.



Let the $\triangle$s ABC, DEF be such that

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}.$$

It is reqd. to prove that the $\triangle$s are similar.

From BA (produced if necessary) cut off BG equal to ED.

" BC " " " BH equal to EF.

Join GH.

Now $\frac{BA}{BG} = \frac{BC}{BH}$ (*Hyp.*), for BG = ED and BH = EF.

$\therefore$ GH is $\parallel$ to AC; (V. 2, *Cor.*)

$\therefore$ $\triangle$s BGH, BAC are similar; (V. 5.)

$$\therefore \frac{BC}{BH} = \frac{CA}{HG}.$$

But $\frac{BC}{EF} = \frac{CA}{FD}$. *Given.*

And BH = EF; $\therefore$ HG = FD.

Also GB = DE. *Constr.*

$\therefore$ the $\triangle$s DEF, GBH are equiangular. (I. 13.)

But $\triangle$s GBH, ABC are equiangular;

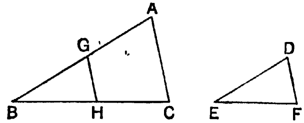
$\therefore$ the $\triangle$s ABC, DEF are equiangular,

and therefore similar. (V. 5.)

Q.E.D.

PROPOSITION 7. THEOREM.

If two triangles have one angle of the one equal to one angle of the other, and the sides about those angles proportional, the triangles are similar.



Let the $\triangle$s ABC, DEF have the $\angle B = \angle E$, and the sides about those $\angle$s proportional, viz. $\frac{AB}{DE} = \frac{BC}{EF}$.

It is reqd. to prove that the $\triangle$*s are similar.*

Apply the $\triangle$DEF to the $\triangle$ABC so that the $\angle$E coincides with the equal $\angle$B, D falling at G in BA, and F falling at H in BC.

Then $\frac{AB}{GB} = \frac{BC}{BH}$. *Given.*

$\therefore$ GH is $\parallel$ to AC; (V. 2, *Cor.*)

$\therefore$ $\triangle$s BGH, BAC are similar; (V. 5.).

i.e. $\triangle$s DEF, ABC are similar. Q.E.D.

EXERCISES.

1. AOB, COD are two straight lines such that AO is to DO as CO is to BO: prove that A, B, C, D are concyclic. LVII. 1.

2. OAB, OCD are two straight lines such that OA is to OC as OD is to OB: prove that the points A, B, D, C are concyclic. LVII. 2.

3. In a triangle ABC, AD is drawn perpendicular to BC. Prove that if AD is a mean proportional between BD and DC, the angle A is a right angle. LVII. 3.

4. D is a point in AB a side of the triangle ABC, and the triangles ADE, ABC on the same side of AB have their angles at D and B equal, and AD is to DE as AB is to BC: prove that AEC is a straight line. LVII. 4.

5. ABC is a triangle and CD is drawn perpendicular to AB; show that if AC is to CD as AB is to BC, then the angle BCD is equal to the angle CAB. LVII. 5.

6. If two triangles ABC, ABD have the angle A common, and BC equal to BD, and if ACD is produced to E so that DE is a third proportional to AC and CB, show that the triangles ABE, ACB are similar. LVII. 6.

7. On opposite sides of the same straight line AB there are described similar isosceles triangles ABC, ABD having BC, AB as their respective bases. From AC any part AE is cut off, and from DA the same proportional part DF. Show that the triangle BEF is similar to each of the given triangles. LVII. 7.

8. ABC is a triangle; D, E, F the middle points of the sides BC, CA, AB. Through D, E, F straight lines are drawn meeting in a point P; and through A, B, C lines are drawn parallel to DP, EP, FP respectively. Prove that these lines also meet in a point. LVII. 8.

9. If two circles cut each other orthogonally, and the line joining their centres meets the circumference of one of them in A and B, prove that if P be any point on the other circle, PA is to PB in a constant ratio. LVII. 9.

10. ABC is an isosceles triangle right-angled at A. Any point P is taken in AB, and BD is drawn at right angles to BC upon the side of BC remote from A, of such length that

BD : BC :: AP : CA.

Prove that CPD is a right angle, and that CP is equal to DP. LVII. 10.

11. ABC is a triangle, D any point in AB produced: E a point in CB, such that CE is to EB as AD is to BD. Prove that DE produced bisects AC. LVII. 11.

12. If D and E are points on the base BC of a triangle ABC such that AB, AC are respectively mean proportionals between BC and BD, BC and CE; show that AD and AE are each a mean proportional between BD and CE. LVII. 12.

13. OAB cuts a circle at A and B, and OC touches it at C. From O, in any direction, OD is drawn equal to OC; AD, BD are joined meeting the circle again at E and F: prove that EF is parallel to OD. LVII. 13.

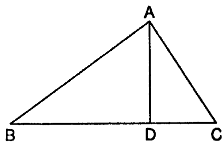
14. C is a point in a given straight line AB, and AB is produced to O, so that CO is a mean proportional between AO and BO. If P be any point on the circle described with centre O and radius OC, prove that the angles APC, BPC are equal. LVII. 14.

15. If the angle A of a triangle ABC and also the adjacent exterior angle be bisected by straight lines, of which the former meets the base in D and the latter the base produced in E, then a straight line drawn through C parallel to AB and terminated by AD produced and AE, is bisected in C. LVII. 15.

Proposition 8. Theorem.

In a right-angled triangle, if a perpendicular be drawn from the right angle to the hypotenuse, the triangles on each side of it are similar to one another and to the whole triangle.

In the $\triangle$ABC, rt.-angled at A, let AD be drawn perp^r to the hypotenuse BC.



It is reqd. to prove that $\triangle$*s* DBA, DAC *are similar to one another and to the whole* $\triangle$ABC.

In the $\triangle$s ABC, DBA,

$\angle$BDA $=$ $\angle$BAC (rt. $\angle$s),

$\angle$B is common;

$\therefore$ the $\triangle$s are equiangular; (I. 7.)

$\therefore$ they are similar. (V. 5.)

In the same way it may be proved that the $\triangle$s DAC, ABC are similar.

Also, since each of the $\triangle$s DBA, DAC is similar to the $\triangle$ABC, they are similar to one another. Q.E.D.

Corollary 1. *The perp^r from the rt. $\angle$ to the base is a mean proportional between the segments of the base.*

For from the similar $\triangle$s BDA, ADC, $\frac{BD}{DA}=\frac{DA}{DC}$.

Corollary 2. *Each side is a mean proportional between the base and the segment of the base adj. to that side.*

E.g. from the similar $\triangle$s BAC, BDA,

$$\frac{BC}{BA}=\frac{BA}{BD}.$$

Note.—We see from the above that

$$DA^2 = \text{rect. } DB \,.\, DC,$$

$$BA^2 = \text{rect. } BD \,.\, BC \quad \text{and} \quad CA^2 = \text{rect. } CD \,.\, CB.$$

Also, we may use these results to prove Pythagoras' Theorem.

For
$$BA^2 = \text{rect. } BD \,.\, BC,$$
$$AC^2 = \text{rect. } CD \,.\, CB.$$
$$\therefore\ BA^2 + AC^2 = \text{rect. } (BD + CD)\, BC$$
$$= BC^2.$$

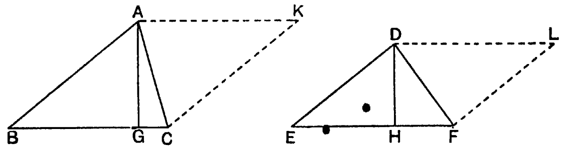
EXERCISES.

1. Prove that either of the equal sides of an isosceles triangle is a mean proportional between the diameter of the circum-circle and the perpendicular from the vertex to the base. LVIII. 1.

2. The hypotenuse of a right-angled triangle is 10, and one side 7 inches long. Find the lengths of the segments of the hypotenuse made by the perpendicular from the right angle. LVIII. 2.

PROPOSITION 9. THEOREM.

If two triangles (or parallelograms) have one angle of the one equal to one angle of the other, their areas are proportional to the areas of the rectangles contained by the sides about the equal angles.



(*a*) Let the $\triangle$s ABC, DEF have $\angle$B equal to $\angle$E.

It is reqd. to prove that $\dfrac{\triangle ABC}{\triangle DEF} = \dfrac{AB \cdot BC}{DE \cdot EF}$.

Draw AG, DH perp^{r} to BC, EF respectively.

Then $$\frac{\triangle ABC}{\triangle DEF} = \frac{\frac{1}{2}AG \cdot BC}{\frac{1}{2}DH \cdot EF} = \frac{AG \cdot BC}{DH \cdot EF} \qquad \text{.........................(1)}$$

In the $\triangle$s ABG, DEH, $\angle B = \angle E$ and $\angle G = \angle H$ (rt. $\angle$s);

$\therefore$ the $\triangle$s are similar; $\therefore$ $\dfrac{AG}{DH} = \dfrac{AB}{DE}$. (V. 5.)

$\therefore$ from (1), $$\frac{\triangle ABC}{\triangle DEF} = \frac{AB \cdot BC}{DE \cdot EF}.$$

(*b*) In the same way, if ABCK, DEFL be two par^{ms} having $\angle B = \angle E$,

$$\frac{\text{par}^{m}\ BK}{\text{par}^{m}\ EL} = \frac{AG \cdot BC}{DH \cdot EF} = \frac{AB \cdot BC}{DE \cdot EF}.$$ Q.E.D.

Trigonometrical proof of Prop. 9.

Area of $\triangle ABC = \frac{1}{2}BC \, . \, AG = \frac{1}{2}BC \, . \, AB \sin B.$

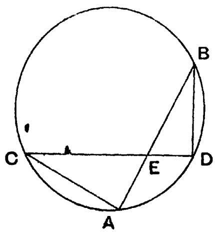
And area of $\triangle DEF = \frac{1}{2}EF \, . \, DH = \frac{1}{2} EF \, . \, DE \sin E$

$= \frac{1}{2}DE \, . \, EF \sin B.$

$$\therefore \frac{\triangle ABC}{\triangle DEF} = \frac{AB \, . \, BC}{DE \, . \, EF}.$$

PROPOSITION 10. THEOREM.

If two chords of a circle cut one another, the rect. contained by the segments of the one is equal to the rect. contained by the segments of the other; and conversely. (See IV. 11.)



Let the chords AB, CD of the circle ABC cut at E.

It is reqd. to prove that the rect. AE . EB = *the rect.* CE . ED.

Join AC and BD.

In the $\triangle$s CEA, BED,

the $\angle$ACE = the $\angle$EBD, in the same segment,

and the $\angle$CEA = the vertically opposite $\angle$BED.

$\therefore$ the $\triangle$s are equiangular;

$$\therefore \frac{CE}{EA} = \frac{BE}{ED}; \qquad \text{(V. 5.)}$$

$\therefore$ the rect. CE . ED = the rect. BE . EA. Q.E.D.

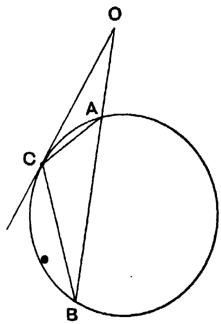
PROPOSITION 11. THEOREM.

If from any point without a circle a secant and a tangent are drawn to the circle, the rect. contained by the whole secant and its external segment is equal to the square on the tangent.

(See IV. 12.)

Let O be a pt. without the circle ABC, and let OAB be a secant, and OC a tangent to the circle.

It is reqd. to prove that the rect. OA . OB = *the sq. on* OC.



In the $\triangle$s OCA, OBC,

$\angle$OCA = $\angle$OBC, in the alt. segment,

and the $\angle$COA is common.

$\therefore$ the $\triangle$s are equiangular;

$$\therefore \frac{OA}{OC} = \frac{OC}{OB};$$

$\therefore$ the rect. OA . OB = OC^2.

Q.E.D.

EXERCISES.

1. ABC, CDE are equal triangles with equal angles at C, and they are on opposite sides of BCE, which is a straight line. Show that a straight line through C, parallel to BD and terminated by AB and DE, is bisected at C. LIX. 1.

2. Two circles whose centres are A, B touch externally at C. A straight line through C meets the first circle again at D, and the second at E. Show that the triangles ACE, BCD are equal in area. LIX. 2.

3. The diameter BCA of the circle whose centre is C, is produced through A to O, and from O the line OPQ is drawn, cutting the circle in P, Q; prove that if the circum-circle of the triangle PCQ meets OC in D, (1) the point D will be fixed for all directions of OPQ; (2) the ratios OA : AD and OC : CP will be equal. LIX. 3.

4. If BFC, CDA, AEB be equilateral triangles described externally on the sides of a triangle ABC right-angled at A; and if AG be drawn perpendicular to BC to meet BC in G; show that the triangles BFG, CFG are respectively equal to the triangles AEB, CDA. LIX. 4.

5. One of the sides containing the right angle of a triangle is double of the other, and circles are described having those two sides as diameters. Show that the length of their common chord is two-fifths of the hypotenuse of the triangle. LIX. 5.

6. In the base AB of a triangle ABC a point D is taken so that the angle CDB is equal to the vertical angle ACB. Prove that CB touches the circum-circle of the triangle ACD. LIX. 6.

7. A straight line meets two intersecting circles in P and Q, R and S, and their common chord in O. Prove that OP, OQ, OR, OS. taken in a certain order, are proportionals. LIX. 7,

8. If ABC be a semi-circle of which O is the centre, and OB perpendicular to AC, and ADE a chord cutting OB in D, prove that the circle described about BDE will be touched by AB. LIX. 8.

9. A and B are two points external to a given circle, C a point on AB such that the tangent from A to the circle is a mean proportional between AB, AC. CD is drawn to touch the circle in D, and AD is drawn cutting the circle again in E. If BE cut the circle again in F, show that DF is parallel to AB. LIX. 9.

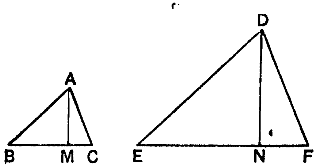
10. If the bisector of the angle A of the triangle ABC meet the base in O and the circum-circle in D, the square on DB is equal to the rectangle DO . DA. LIX. 10.

11. AOB, COD are two straight lines such that AO is to OB as 3 is to 4, and CO is to OD as 2 is to 5: compare the areas of the triangles AOC, BOD. Also find the ratio of the areas of the triangles AOD, BOC. LIX. 11.

12. AOC, BOD are two triangles of equal area, AOB, COD being straight lines. If AO = 4, BO = 2, CO = 5, find the length of DO. LIX. 12.

Proposition 12. Theorem.

The areas of similar triangles are proportional to the squares on their corresponding (or homologous) sides.



Let ABC, DEF be two similar Δs having the $\angle$s at A, B, C respectively equal to the $\angle$s at D, E, F.

It is reqd. to prove that

$$\frac{\triangle ABC}{\triangle DEF}=\frac{AB^2}{DE^2}=\frac{BC^2}{EF^2}=\frac{AC^2}{DF^2}.$$

Draw AM perp^r to BC and DN perp^r to EF.

$$\frac{\triangle ABC}{\triangle DEF}=\frac{\frac{1}{2}AM \,.\, BC}{\frac{1}{2}DN \,.\, EF}. \quad\text{..............................(1)}$$

But $\frac{AM}{DN}=\frac{AB}{DE}$ (for the $\triangle$s ABM, DEN are equiangular to one another)

$=\frac{BC}{EF}$ (for the $\triangle$s ABC, DEF are similar);

$\therefore$ from (1), $$\frac{\triangle ABC}{\triangle DEF}=\frac{BC^2}{EF^2}$$

$$=\frac{AB^2}{DE^2}=\frac{AC^2}{DF^2}\left(\text{for } \frac{AB}{DE}=\frac{BC}{EF}=\frac{AC}{DF}, \textit{ Hyp.}\right).$$ Q.E.D.

NOTE.—Euclid enunciated this proposition in the following manner: *Similar triangles are to one another in the duplicate ratio of their homologous sides.* See paragraph 6 on Ratio, where it is explained that the duplicate ratio of AB to DE is the same as the ratio of AB^2 to DE^2.

EXERCISES.

1. Two corresponding sides of two similar triangles are respectively 5 and 3 inches long; the area of the first triangle is 75 sq. in.: find the area of the second. LX. 1.

2. In a triangle ABC, DE is drawn parallel to the base BC and cuts the side AB in the ratio of 3 to 5: compare the areas of the triangles ADE, ABC. LX. 2.

3. ABC is a given triangle: draw a straight line parallel to BC making with AB and AC a triangle equal to four times the given triangle. LX. 3.

4. Draw a straight line parallel to the base BC of a given triangle ABC making with AB and AC a triangle equal to nine times the given triangle. LX. 4.

5. DE is drawn parallel to the base BC of a triangle ABC, dividing the sides AB, AC in the ratio of 1 to 3. Find the ratio of the areas of the triangles ADE, ABC. LX. 5.

6. DE is drawn parallel to the base BC of a triangle ABC meeting AB, AC in D and E, so that the triangle ADE is one-ninth of the whole triangle: find the ratio of AD to DB. LX. 6.

7. If AD, BE are the perpendiculars on the sides BC, AC of the triangle ABC, prove that the triangles CDE, ABC are in the ratio of the squares on CD and CA. LX. 7.

8. D, E, F are the feet of the perpendiculars from the angles A, B, C of an acute-angled triangle on the opposite sides; and G, H the feet of perpendiculars from E, F on BC. Prove that the straight lines DE, DF divide the triangle ABC into three parts which are proportional to CG, GH and HB. LX. 8.

9. The angle BAC of the triangle ABC is bisected by AD which cuts the base BC in D, and DE, DF are drawn respectively parallel to AB, AC to cut the sides AC, AB in E, F. Show that BF is to CE in the ratio of the squares on AB and AC. LX. 9.

10. Any triangle described on a side of a square as base is one-half the similar triangle described on a diagonal as base. LX. 10.

11. Through the vertices A, B, C of an equilateral triangle straight lines are drawn perpendicular to the sides AB, BC, CA respectively, so as to form another equilateral triangle. Compare the areas of the two triangles. LX. 11.

12. Through the ends of the base BC of an equilateral triangle ABC straight lines BD, CD are drawn respectively perpendicular to the sides AB, AC. Prove that the area of the triangle BCD is one-third of the area of the triangle ABC. LX. 12.

13. If ABC be a triangle, and D a point on BC such that BC is equal to the diagonal of the square on DC; show that a line through D parallel to AB bisects the triangle. LX. 13.

14. If AB, CD be two chords of a circle intersecting in an external point E, show that the triangle EAC has to the triangle EBD the duplicate ratio of that which AC has to BD. LX. 14.

15. In a right-angled triangle, if a perpendicular be drawn from the right angle to the base, the segments of the base are to one another in the duplicate ratio of the sides which contain the right angle. LX. 15.

16. ABC is a triangle with an obtuse angle at A; and AM, AN are drawn to meet the base BC in M and N respectively, so that the angles AMB, ANC are each equal to BAC; show that BM has to NC the duplicate ratio of that which AB has to AC. LX. 16.

17. If AB, BC be two sides of a regular figure; L and M their respective middle points; and O the centre of the inscribed circle; show that the triangle BLM has to the triangle OLM the duplicate ratio of that which a side of the figure has to the diameter of the circle. LX. 17.

18. If C be the centre of a circle, OPCQ a straight line cutting the circle in P and Q, OT a tangent to the circle, and PN another tangent cutting OT in N; show that the triangles OPN and OTC bear the same ratio to one another that OP bears to OQ. LX. 18.

19. If the sides AB, BC, CD, DA of a quadrilateral be trisected respectively in E and F, G and H, K and L, M and N; show that the figure EFGHKLMN is seven-ninths of the whole quadrilateral. LX. 19.

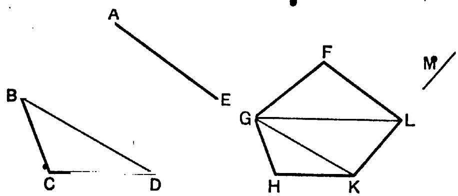
20. Of two regular hexagons, one is inscribed in a circle and the other is described about the same circle: prove that the areas of the hexagons are to one another in the ratio of 3 to 4. LX. 20.

21. ADOB is the diameter of a circle whose centre is O; C is a point on the circumference such that CD is perpendicular to AB; and EC, EA are tangents to the circle; show that

$$\triangle ECA : \triangle OCB :: AD : DB. \qquad \text{LX. 21.}$$

PROPOSITION 13. THEOREM.

Similar rectilineal figures may be divided into the same number of similar triangles by joining corresponding angles: these triangles have the same ratio to one another that the figures have; and the figures are to one another in the ratio of the squares on the corresponding or homologous sides.



Let ABCDE, FGHKL be similar rectil. figs., AB and FG being corresponding or homologous sides, and A and F corresponding angles.

(1) *To prove that the figures may be divided into the same number of similar triangles.*

Join BE, BD, GL, GK.

The figures are similar;

$$\therefore \angle A = \angle F,$$

and
$$\frac{BA}{GF} = \frac{AE}{FL};$$

$\therefore$ $\triangle$s BAE, GFL are similar, and $\angle AEB = \angle FLG$. (V. 7.)

But $\angle AED = \angle FLK$, for the figs. are similar;

$\therefore$ the remaining $\angle BED$ = the remaining $\angle GLK$.

Also the $\triangle$s ABE, FGL have been proved similar.

$$\therefore \frac{BE}{GL} = \frac{AE}{FL} = \frac{ED}{LK}, \text{ for the figs. are similar,}$$

and $\angle BED$ has been proved equal to $\angle GLK$;

$\therefore$ the $\triangle$s BED, GLK are similar. (V. 7.)

In the same way it may be proved that the $\triangle$s BCD, GHK are similar.

$\therefore$ the figures are divided into the same number of similar $\triangle$s.

(2) *To prove that the $\triangle$s have to one another the same ratio that the figures have.*

$\triangle$s ABE, FGL are similar;

$$\therefore \frac{\triangle ABE}{\triangle FGL} = \frac{BE^2}{GL^2} \qquad \text{(V. 12.)}$$

$$= \frac{\triangle BDE}{\triangle GKL}, \text{ for these } \triangle\text{s are similar,} \qquad ,,$$

$$= \frac{BD^2}{GK^2} = \frac{\triangle BCD}{\triangle GHK}. \qquad ,, \qquad ,, \qquad ,,$$

$$\therefore \frac{\triangle ABE}{\triangle FGL} = \frac{\triangle BDE}{\triangle GKL} = \frac{\triangle BCD}{\triangle GHK}$$

$$= \frac{\triangle ABE + \triangle BDE + \triangle BCD}{\triangle FGL + \triangle GKL + \triangle GHK}$$

$$= \frac{\text{fig. ABCDE}}{\text{fig. FGHKL}};$$

i.e. the triangles have to one another the same ratio that the figs. have.

(3) *To prove that the figs. are in the ratio of the sqs. on the corresponding sides* AB, FG.

$$\text{For } \frac{\text{fig. ABCDE}}{\text{fig. FGHKL}} = \frac{\triangle ABE}{\triangle FGL} \qquad \textit{Proved in Part (2).}$$

$$= \frac{AB^2}{FG^2}, \text{ for } \triangle\text{s ABE, FGL are similar.} \qquad \text{(V. 12.)}$$

COROLLARY. *If three straight lines are proportionals, the* **first** *is to the* **third** *as any rectilineal figure described on the* **first** *is to a similar and similarly situated rectilineal figure described on the* **second.**

Let M be a third proportional to AB and FG; then

$$\frac{AB}{FG}=\frac{FG}{M};\quad \therefore\ \frac{AB}{M}=\frac{AB^2}{FG^2}\quad \text{(Paragraph 6, on Ratio.)}$$

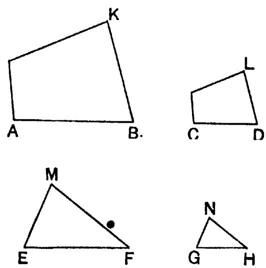
$$=\frac{\text{polygon ABCDE}}{\text{polygon FGHKL}}.\quad \text{(For the polygons are similar.)}$$

Q.E.D.

PROPOSITION 14. THEOREM.

If four straight lines are proportional, and a pair of similar rectilineal figures are similarly described on the first and second, and also a pair of similar rectilineal figures are similarly described on the third and fourth, the figures are proportional.

Conversely, if similar and similarly described rectilineal figures on the first and second of four straight lines are proportional to similar and similarly described rectilineal figures on the third and fourth, the four straight lines are proportional.



(1) Let the str. lines AB, CD, EF, GH be proportional, and let ABK, CDL be similar and similarly described rectil. figs. on AB, CD; and let EFM, GHN be like rectil. figs. on EF, GH.

It is reqd. to prove that

$$\frac{\text{fig. ABK}}{\text{fig. CDL}}=\frac{\text{fig. EFM}}{\text{fig. GHN}}.$$

The figs. ABK, CDL are similar;

$$\therefore \frac{\text{fig. ABK}}{\text{fig. CDL}} = \frac{AB^2}{CD^2}.$$

In the same way, $$\frac{\text{fig. EFM}}{\text{fig. GHN}} = \frac{EF^2}{GH^2}.$$

But by hyp., $$\frac{AB}{CD} = \frac{EF}{GH};$$

$$\therefore \frac{AB^2}{CD^2} = \frac{EF^2}{GH^2};$$

$$\therefore \frac{\text{fig. ABK}}{\text{fig. CDL}} = \frac{\text{fig. EFM}}{\text{fig. GHN}}.$$

(2) Let the similar and similarly described figs. on AB and CD be proportional to like figs. on EF, GH.

It is reqd. to prove that

$$\frac{AB}{CD} = \frac{EF}{GH}.$$

As in Part (1), $$\frac{\text{fig. ABK}}{\text{fig. CDL}} = \frac{AB^2}{CD^2},$$

and $$\frac{\text{fig. EFM}}{\text{fig. GHN}} = \frac{EF^2}{GH^2}.$$

But $$\frac{\text{fig. ABK}}{\text{fig. CDL}} = \frac{\text{fig. EFM}}{\text{fig. GHN}};$$ *Given.*

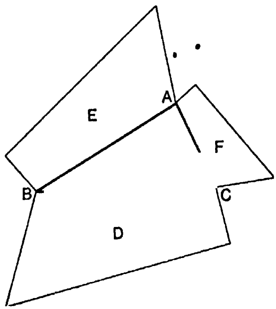
$$\therefore \frac{AB^2}{CD^2} = \frac{EF^2}{GH^2};$$

$$\therefore \frac{AB}{CD} = \frac{EF}{GH}.$$

Q.E.D.

PROPOSITION 15. THEOREM.

In a right-angled triangle, any rectilineal figure described on the hypotenuse is equal to the sum of the similar and similarly described figures on the other two sides.



Let $\triangle ABC$ be rt. angled at A, and let D, E, F be similar and similarly described rectil. figs. on BC, AB, CA.

It is reqd. to prove that fig. D = fig. E + fig. F.

The figs. D and E are similar;

$$\therefore \frac{\text{fig. E}}{\text{fig. D}} = \frac{AB^2}{BC^2}. \qquad \text{(V. 13.)}$$

In the same way, $$\frac{\text{fig. F}}{\text{fig. D}} = \frac{AC^2}{BC^2};$$

$$\therefore \frac{\text{fig. E} + \text{fig. F}}{\text{fig. D}} = \frac{AB^2 + AC^2}{BC^2}.$$

But $AB^2 + AC^2 = BC^2$ for $\angle A$ is a rt. $\angle$; (II. 16.)

$\therefore$ fig. E + fig. F = fig. D. Q.E.D.

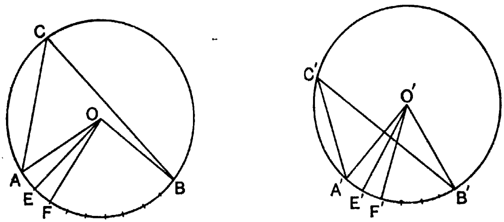
NOTE.—This prop. generalizes the result of II. 16.

EXERCISE.

ABCD is a parallelogram whose side AB is divided at E so that AE is less than EB. CF (equal to AE) is drawn perpendicular to DC on the side remote from AB; and FG (equal to EB) is drawn to meet DC produced at G. If GE and DA meet, when produced, in H, show that the triangle HDG is equal to the parallelogram ABCD. LXI. 1.

Proposition 16. Theorem.

In equal circles, angles at the centres, or angles at the circumferences, have the same ratio as the arcs on which they stand; so also have the sectors.



Let ABC, A'B'C' be equal circles, and let AOB, A'O'B' be angles at the centres.

(1) *It is reqd. to prove that* $\dfrac{\angle AOB}{\angle A'O'B'} = \dfrac{\text{arc } AB}{\text{arc } A'B'}$.

Let arc AB be p units, and arc A'B' q units of length. Take arcs AE, EF, ..., A'E', E'F', ... each one unit of length, and join OE, O'E'.

Equal arcs subtend equal $\angle$s;

$$\therefore \angle AOB = p \text{ times } \angle AOE;$$

and $$\angle A'O'B' = q \text{ times } \angle A'O'E';$$

$$\therefore \frac{\angle AOB}{\angle A'O'B'} = \frac{p}{q}, \text{ for } \angle AOE = \angle A'O'E'.$$

Also $$\text{arc } AB = p \text{ times arc } AE,$$

and $$\text{arc } A'B' = q \text{ times arc } A'E';$$

$$\therefore \frac{\text{arc } AB}{\text{arc } A'B'} = \frac{p}{q}$$

$$= \frac{\angle AOB}{\angle A'O'B'}, \qquad \textit{Proved above.}$$

i.e. the $\angle$s at the centres are proportional to the arcs on which they stand.

(2) Let ACB, A′C′B′ be ∠s at the circumferences.

It is reqd. to prove that $\dfrac{\angle ACB}{\angle A'C'B'} = \dfrac{\text{arc } AB}{\text{arc } A'B'}$.

∠AOB at the centre = twice ∠ACB at the circumference, and ∠A′O′B′ at the centre = twice ∠A′C′B′ at the circumference; (III. 4.)

$$\therefore \frac{\angle ACB}{\angle A'C'B'} = \frac{\angle AOB}{\angle A'O'B'}$$

$$= \frac{\text{arc } AB}{\text{arc } A'B'}, \quad \textit{Proved in Part (1).}$$

i.e. the ∠s at the circumferences are proportional to the arcs on which they stand.

(3) *It is reqd. to prove that*

$$\frac{\text{arc } AB}{\text{arc } A'B'} = \frac{\text{sector } AOB}{\text{sector } A'O'B'}.$$

Equal arcs subtend equal ∠s; ∴ ∠AOE = ∠EOF and so on;

∴ by superposition, sector AOE = sector EOF;

∴ sector AOB = p times sector AOE.

In the same way, sector A′O′B′ = q times sector A′O′E′.

Also by superposition, sector AOE = sector A′O′E′;

$$\therefore \frac{\text{sector } AOB}{\text{sector } A'O'B'} = \frac{p}{q} = \frac{\text{arc } AB}{\text{arc } A'B'},$$

i.e. in equal circles sectors are proportional to their arcs.

Q.E.D.

This proposition is true for one circle.

APPLICATION TO TRIGONOMETRY.

The angle subtended at the centre of a circle by an arc equal to the radius is called a radian. Hence if arc AB subtend an angle of θ radians,

$$\frac{\text{arc AB}}{r} = \frac{\theta}{1},$$

or

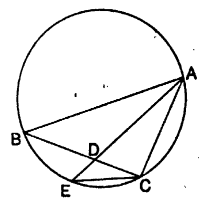
$$\text{arc AB} = r\theta.$$

EXERCISE.

Two equal circles are such that the tangents at either of their points of intersection are at right angles. Show that the area common to the two circles together with the square on the radius is equal to half the area of either circle. LXII. 1.

PROPOSITION 17. THEOREM.

If the vertical angle of a triangle be bisected by a straight line which cuts the base, the square on this bisector is equal to the difference between the rectangle contained by the sides of the triangle and the rectangle contained by the segments of the base.



Let ABC be a $\triangle$ having the vertical $\angle$A bisected by AD which cuts the base at D.

It is reqd. to prove that

$$AD^2 = rect.\ AB \cdot AC - rect.\ BD \cdot DC.$$

Let ABEC be the circle circumscribing the $\triangle$ABC, and let AD produced meet it at E.

Join EC.

In the $\triangle$s BDA, ECA, $\angle BAD = \angle EAC$, *Given.*

$\angle DBA = \angle CEA$ in the same segment;

$\therefore$ the $\triangle$s are equiangular, (I. 7.)

and consequently similar; (V. 5.)

$$\therefore \frac{BA}{AE} = \frac{AD}{AC};$$

$\therefore$ rect. AB . AC = rect. AE . AD

= rect. AD (DE + DA)

= rect. AD . DE + AD^2. (IV. 3.)

But BC, AE are chords of the circle;

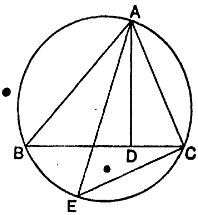
$\therefore$ rect. AD . DE = rect. BD . DC; (IV. 11.) or (V. 10.)

$\therefore$ rect. AB . AC = rect. BD . DC + AD^2,

i.e. AD^2 = rect. AB . AC − rect. BD . DC. Q.E.D.

Proposition 18. Theorem.

If from the vertical angle of a triangle a perpendicular be drawn to the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by the perpendicular and the diameter of the circle circumscribing the triangle.



Let ABC be a $\triangle$, AD the perp^r from the pt. A to the base BC, and AE a diameter of the circle circumscribing the $\triangle$.

It is reqd. to prove that

the rect. AB . AC = *rect.* AD . AE.

Join EC.

In the $\triangle$s ABD, AEC,

$\angle$ABD = $\angle$AEC, $\angle$s in the same segment,

rt. $\angle$ADB = rt. $\angle$ACE, the $\angle$ in a semi-circle;

$\therefore$ the $\triangle$s are equiangular; (I. 7.)

$\therefore$ they are also similar; (V. 5.)

$$\therefore \frac{AB}{AE} = \frac{AD}{AC};$$

$\therefore$ rect. AB . AC = rect. AD . AE. Q.E.D.

Trigonometrical Proof.

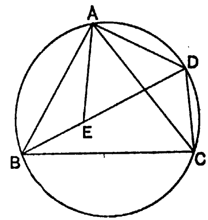
With the usual notation, R denoting the rad. of the circum-circle,

since ECA is a rt. $\angle$, $\dfrac{b}{2R} = \sin E = \sin B = \dfrac{AD}{c}$;

$\therefore$ $bc = 2R . AD$. Q.E.D.

PROPOSITION 19. PTOLEMY'S THEOREM.

The rectangle contained by the diagonals of a cyclic quadrilateral is equal to the sum of the two rectangles contained by its opposite sides.



Let ABCD be a quad[l] inscribed in a circle, and AC, BD its diagonals.

It is reqd. to prove that

rect. AC . BD = *rect.* AD . BC + *rect.* AB . CD.

Make the ∠BAE equal to the ∠CAD. (I. 25.)

Add ∠EAC to each; then

∠BAC = ∠EAD.

Then in △s ABC, AED, ∠BAC = ∠EAD,

∠BCA = ∠EDA, in the same segment; (III. 5.)

∴ the △s are equiangular; (I. 7.)

∴ they are also similar; (V. 5.)

$$\therefore \frac{BC}{AC} = \frac{ED}{AD};$$

rect. AD . BC = rect. AC . ED. ....(1)

Again, in the △s ABE, ACD, ∠BAE = ∠CAD, *Constr.*

∠ABE = ∠ACD, in the same segment;

∴ the △s are equiangular; (I. 7.)

∴ they are similar; (V. 5.)

$$\therefore \frac{AB}{BE} = \frac{AC}{CD};$$

∴ rect. AB . CD = rect. AC . BE;

∴ from (1), rect. AD . BC + rect. AB . CD

= rect. AC . ED + rect. AC . BE

= rect. AC . BD. (IV. 1.)

Q.E.D.

EXERCISES.

1. If the angle A of the triangle ABC is bisected by ADE meeting the base in D and the circum-circle in E, and if AD is a mean proportional between BD and DC, prove that the square on AE is twice the square on CE. LXIII. 1.

2. ADE is a straight line which divides the base BC of a triangle ABC so that BD is to DC as BA is to AC, and which cuts the circum-circle of the triangle in E. Show that the rectangle AB . AC is equal to the difference of the squares on AE, BE. LXIII. 2.

3. ABCD is a quadrilateral inscribed in a circle. Find a point P in the circumference such that the rectangle PA . PC shall be equal to the rectangle PB . PD. LXIII. 3.

4. If P is a point on the arc BC of the circle circumscribing an equilateral triangle ABC, PB + PC = PA. LXIII. 4.

5. If the part of the bisector of the vertical angle of a triangle cut off by the circum-circle is bisected by the base, show that the square on each side is twice the square on the segment of the base adjacent to it. LXIII. 5.

6. The base of a triangle is given in magnitude and position, and the rectangle contained by the sides varies as the area of the triangle: find the locus of the vertex. LXIII. 6.

7. Show that the rectangle contained by two adjacent sides of a quadrilateral inscribed in a circle is to the rectangle contained by the other two sides in the ratio of the segments into which one of the diagonals of the quadrilateral is divided by the other. LXIII. 7.

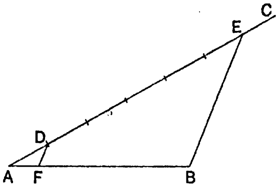
8. If the exterior angle CBD of a triangle ABC is bisected by a straight line, which cuts the base produced in E, prove that the square on BE is equal to the difference of the rectangles contained by AE, EC, and AB, BC. LXIII. 8.

9. In a quadrilateral inscribed in a circle, one of the diagonals is a diameter of the circle and bisects the other: find the relation between the sum of the rectangles contained by the opposite sides of the quadrilateral and the area of the quadrilateral. LXIII. 9.

10. If ABC be a triangle inscribed in a circle, and AD be drawn touching the circle to meet BC produced in D, the diameters of the circles about ABD and ACD will be as AD to CD. LXIII. 10.

Proposition 20. Problem.

From a given straight line to cut off any assigned submultiple.



Let AB be the given str. line.

It is reqd. to find a pt. F *in* AB *such that* AB $= p$ *times* AF, *where* p *is a given integer.*

From A draw AC making any $\angle$ with AB, and in it take any point D.

By cutting off from AC successive lengths each equal to AD, make AE equal to p times AD.

Join BE, and from D draw DF parallel to EB, meeting AB at F.

F *will be the reqd. pt.*

In the $\triangle$ABE, DF is parallel to EB;

$$\therefore \frac{AB}{AF} = \frac{AE}{AD} \qquad \text{(V. 2, } Cor.\text{)}$$

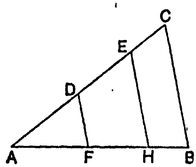
$$= p;$$

$\therefore$ AB $= p$ times AF;

$\therefore$ F is the reqd. pt.

Proposition 21. Problem.

To divide a straight line similarly to a given divided straight line.



Let AB be the given str. line to be divided, and AC the given str. line divided at D and E.

It is reqd. to divide AB *similarly to* AC.

Let AB and AC be placed to contain any angle.

Join BC, and through D draw DF par[l] to CB, to meet AB at F.

Also through E draw EH par[l] to CB, to meet AB at H.

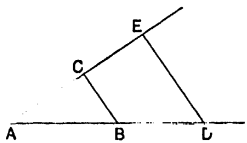
AB *will be divided at* F *and* H *similarly to* AC.

Proof. Use the method of V. 2, Cor. 2.

The same construction applies if AC is divided into more than three parts.

PROPOSITION 22. PROBLEM.

To find a third proportional to two given straight lines.



Let AB, AC be the given str. lines.

It is reqd. to find a third proportional to AB *and* AC.

Place AB and AC so as to form any angle. Join CB.

In AB produced make BD equal to AC, and from D draw DE par[l] to BC to meet AC produced at E. (I. 26.)

CE *will be the third proportional reqd.*

In the $\triangle$ADE, BC is par[l] to DE;

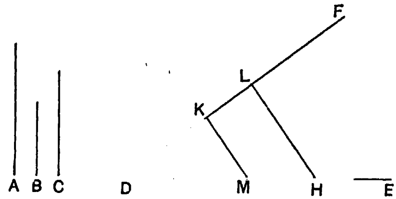
$$\therefore \frac{AB}{BD} = \frac{AC}{CE}. \qquad (V. 2.)$$

But BD = AC; $\quad \therefore \dfrac{AB}{AC} = \dfrac{AC}{CE},$

i.e. CE is a third proportional to AB and AC. Q.E.F.

PROPOSITION 23. PROBLEM.

To find a fourth proportional to three given straight lines.



Let A, B, C be the three given str. lines.

It is reqd. to find a fourth proportional to A, B, C.

Draw two str. lines DE, DF of indefinite length forming any angle.

From DE cut off DM equal to A, and MH equal to B.

From DF cut off DK equal to C.

Join KM, and from H draw HL || to MK to meet DF at L.

KL *will be the fourth proportional reqd.*

In the $\triangle$DLH, KM is || to LH;

$$\therefore \frac{DM}{MH} = \frac{DK}{KL}. \qquad \text{(V. 2.)}$$

But DM = A, MH = B, and DK = C;

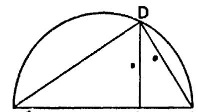
$$\therefore \frac{A}{B} = \frac{C}{KL},$$

i.e. KL is a fourth proportional to A, B, and C.

Q.E.F.

PROPOSITION 24. PROBLEM.

To find a mean proportional to two given straight lines.



Let AB, BC be the given str. lines.

It is reqd. to find a mean proportional to AB *and* BC.

Place AB and BC in the same str. line, and on AC describe the semi-circle ADC.

From B draw BD at rt. ∠s to AC to meet the semi-circle at D. (I. 22.)

BD *will be the mean proportional reqd.*

Join AD, DC.

ADC is a semi-circle; ∴ the ∠ADC is a rt. ∠. (III. 7.)

Then in the rt. angled △ADC, DB is drawn from the rt. ∠ perp[r] to the hypotenuse;

∴ the △s ABD, DBC are similar; (V. 8.)

$$\therefore \frac{AB}{BD} = \frac{BD}{BC},$$

i.e. BD is a mean proportional between AB and BC.

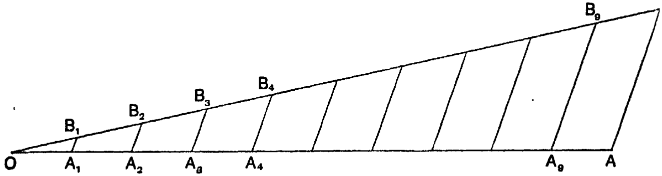
Q.E.F.

DIAGONAL SCALE.

By means of a Diagonal Scale, with the help of dividers, we can measure lengths to the nearest hundredth part of an inch.

Its construction depends upon the following. If OA and OB are two straight lines and OA is divided into 10 equal parts at A_1, A_2, etc., and parallels A_1B_1, A_2B_2, ... to AB are drawn, then

$$A_1B_1 = \tfrac{1}{10}AB, \quad A_2B_2 = \tfrac{2}{10}AB, \quad \text{and so on.}$$



Thus if AB were equal to one-tenth of an inch,

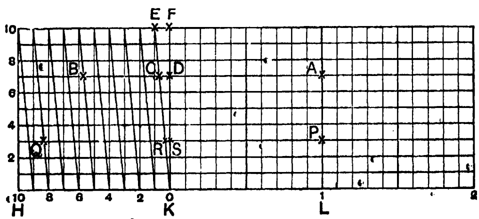
A_1B_1 would be equal to $\frac{1}{100}$th of an inch.

A_2B_2 „ „ $\frac{2}{100}$ths „

A_9B_9 „ „ $\frac{9}{100}$ths „

The next figure shows one end of a Diagonal Scale. It is generally drawn on hard wood, or ivory.

HK = KL = 1 inch.



Each of the small divisions along the top and bottom lines = one-tenth of an inch. Thus EF = $\frac{1}{10}$th of an inch.

Examining the portion KEF, we see, from what has been said above, that

$$RS = \tfrac{3}{10}EF = \tfrac{3}{100}\text{ths of an inch.}$$

Also $$CD = \tfrac{7}{10}EF = \tfrac{7}{100}\text{ths} \quad ,, \quad ,,$$

$$\therefore PR = PS + SR = 1 + \tfrac{3}{100} = 1{\cdot}03 \text{ in.}$$

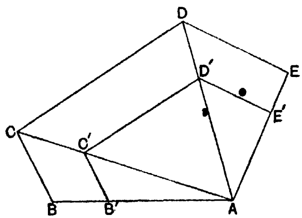
and $$AC = AD + DC = 1 + \tfrac{7}{100} = 1{\cdot}07 \text{ in.}$$

Again $$AB = AD + CB + DC = 1 + \tfrac{5}{10} + \tfrac{7}{100} = 1{\cdot}57 \text{ in.}$$

and $$PQ = PS + RQ + SR = 1 + \tfrac{8}{10} + \tfrac{3}{100} = 1{\cdot}83 \text{ in.}$$

PROPOSITION 25. THEOREM.

On a given straight line to describe a rectilineal figure similar to a given rectilineal figure.



Let AB′ be the given str. line, and ABCDE the given rectil. fig. Take AB′ along AB.

It is reqd. to describe on AB′ *a rectil. fig. similar to the fig.* ABCDE.

Join AC, AD.

Draw B′C′ parallel to BC cutting AC at C′.

,, C′D′ ,, CD ,, AD ,, D′.

,, D′E′ ,, DE ,, AE ,, E′.

AB′C′D′E′ *will be the reqd. fig.*

The ∠AB′C′ = the ∠ABC, for B′C′ is ∥ to BC.

The ∠B′C′A = the ∠BCA, ,, ,,

and the ∠AC′D′ = the ∠ACD, ,, C′D′ ,, CD.

∴ the ∠B′C′D′ = the ∠BCD.

In the same way, the $\angle C'D'E'$ = the $\angle CDE$,

and the $\angle D'E'A$ = the $\angle DEA$.

$\therefore$ the fig. $AB'C'D'E'$ is equiangular to the fig. $ABCDE$.

Also $\dfrac{AB'}{AB} = \dfrac{B'C'}{BC}$, for the $\triangle$s $AB'C'$, ABC are equiangular,

$= \dfrac{AC'}{AC}$ „ „ „ „

$= \dfrac{C'D'}{CD}$ „ $\triangle$s $AC'D'$, ACD „ „

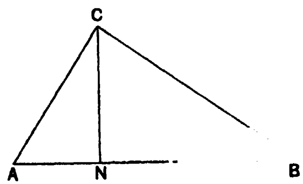
$= \dfrac{D'E'}{DE} = \dfrac{AE'}{AE}$ in the same way.

i.e. the figs. $AB'C'D'E'$, $ABCDE$ have their sides taken in order about their equal $\angle$s proportional.

$\therefore$ the fig. $AB'C'D'E'$ is similar to the fig. $ABCDE$.

Q.E.F.

The following are very useful in solving many problems.



Take ABC, a $\triangle$ rt. angled at C, and draw CN perp^r to the base AB.

If $AN = x$ inches. (Any unit may be used.)

and $BN = 1$ in.,

$CN^2 = AN \cdot BN = x$ sq. in. (V. 8, Note.)

$\therefore$ $CN = \sqrt{x}$ in.

Again, if $AN = 1$ in. and $AB = x$ in.,

$AC^2 = AB \cdot AN = x$ sq. in. (V. 8, Note.)

$\therefore$ $AC = \sqrt{x}$ in.

Ex. *Bisect a given* $\triangle ABC$ *by a line drawn parallel to* BC.

We have to find a $\triangle ADE$ such that, DE being $\parallel$ to BC,

$$\frac{\triangle ADE}{\triangle ABC}=\frac{1}{2}, \quad i.e. \quad \frac{AD^2}{AB^2}=\frac{1}{2} \quad \text{and} \quad \frac{AD}{AB}=\frac{1}{\sqrt{2}}.$$

Hence the following construction.

On AB describe a semi-circle, and draw OF perp^r to AB from the centre O.

From AB cut off AD equal to AF, and draw DE $\parallel$ to BC.

$$\frac{\triangle ADE}{\triangle ABC}=\frac{AD^2}{AB^2}=\frac{AF^2}{AB^2}=\frac{AO\,.\,AB}{AB^2}=\frac{AO}{AB}=\frac{1}{2}.$$

$\therefore$ $\triangle ADE=\frac{1}{2}\triangle ABC$, and DE is $\parallel$ to BC. Q.E.F.

Proposition 26. Problem.

To describe a square equal in area to a given rectilineal figure.

(1) Reduce the rectil. fig. to a $\triangle$.

(2) This $\triangle$ is equal to a rect. whose sides are the base of the $\triangle$ and half its altitude.

(3) Use either of the following methods to obtain the side of the square.

In the fig. p. 226, if AN, BN are the sides of the rect.,

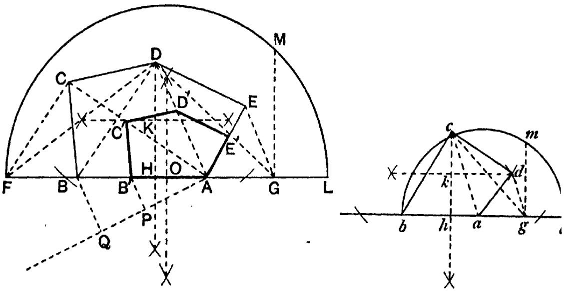
$CN^2=AN\,.\,BN$, and CN is a side of the sq.

If AN, AB are the sides of the rect.,

$AC^2=AN\,.\,AB$, and AC is a side of the sq.

PROPOSITION 27. PROBLEM.

To describe a rectilineal figure which shall be similar to one and equal to another rectilineal figure.



Let ABCDE and *abcd* be the two given rectil. figs.

It is reqd. to describe a fig. equal to abcd and similar to ABCDE.

Reduce the fig. ABCDE to an equal $\triangle$DFG. (As in II. 21.)

Draw DH perp^r to the base FG, and bisect it at K.

Along FG produced make GL equal to HK, and bisect FL at O.

With centre O and rad. OF, or OL, describe a semi-circle FML, and at G draw GM ‖ to DH (*i.e.* perp^r to FL) to meet the circle at M.

In the same way, reduce the fig. *abcd* to an equal $\triangle$*cbg*, and in *bg* produced make *gl* equal to *hk*, half the altitude of the $\triangle$*cbg*.

On *bl* describe a semi-circle *bml*, and draw *gm* ‖ to *ch* to meet the circle at *m*.

Thro. the pt. A draw any str. line APQ making AP equal to *gm* and AQ equal to GM.

Join QB, and draw PB′ ‖ to it to meet AB at B′.

On AB′ describe the fig. AB′C′D′E′ similar to ABCDE. (V. 25.)

AB′C′D′E′ *is the reqd. fig.*

GM^2 = the rect. FG . GL (V. 8, Note,) = $\frac{1}{2}$DH . FG

= the $\triangle$DFG = fig. ABCDE.

In the same way, $gm^2 = \text{fig. } abcd.$

$$\frac{\text{Fig. AB'C'D'E'}}{\text{fig. ABCDE}} = \frac{\text{AB'}^2}{\text{AB}^2} \text{ (V. 13) } = \frac{\text{AP}^2}{\text{AQ}^2} \text{ (V. 2) } = \frac{gm^2}{\text{GM}^2}$$

$$= \frac{\text{fig. } abcd}{\text{fig. ABCDE}}.$$

∴ fig. AB′C′D′E′ = fig. *abcd*, and it is similar to fig. ABCDE;

∴ AB′C′D′E′ is the reqd. fig.

N.B.—All the lines of construction are shown in the diagrams.

Proposition 28. Problem.

To describe a figure similar to the given figure ABCDEF *and n times its area.*

Draw LG n in. long, and on it describe the semi-circle LKG.

Make LH 1 in. long, and draw HK perp[r] to LG to meet the circle at K.

From A draw any str. line APQ, making AP = 1 in. and AQ = LK.

Join PB, and draw Qb ∥ to PB to meet AB produced at b.

On Ab describe the fig. A$bcdef$ similar to the given fig.

A$bcdef$ *will be the fig. reqd.*

For
$$\frac{\text{Fig. A}bcdef}{\text{fig. ABCDEF}} = \frac{\text{A}b^2}{\text{AB}^2} \text{ (V. 13) } = \frac{\text{AQ}^2}{\text{AP}^2} \text{ (V. 2)}$$

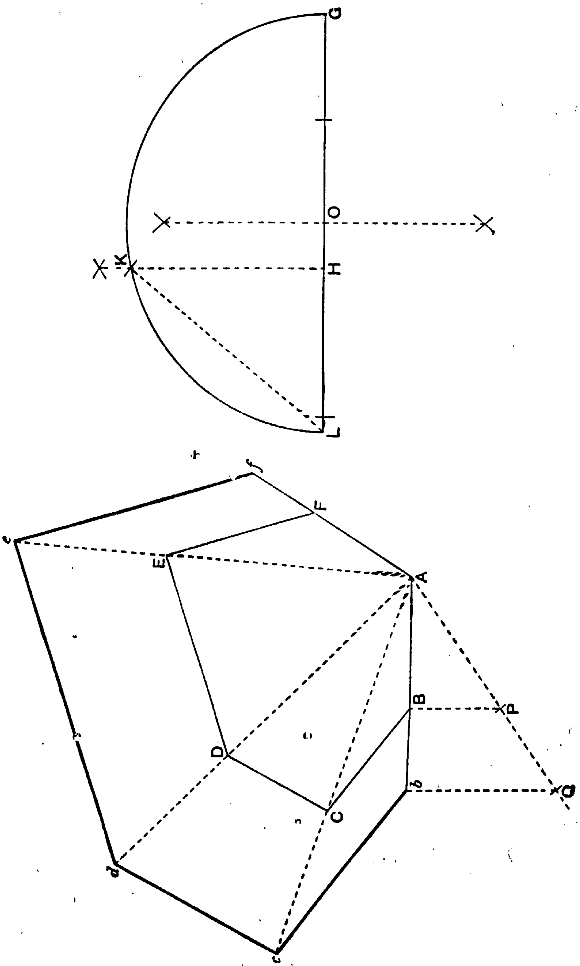
$$= \frac{\text{LK}^2}{\text{LH}^2} = \frac{\text{LH . LG}}{\text{LH}^2} \qquad \text{(V. 8, Note)}$$

$$= \frac{\text{LG}}{\text{LH}} = n;$$

∴ fig. A$bcdef$ is the fig. reqd.

N.B.—**All the lines of construction are shown in the diagrams.**
If n is a fraction, say $\frac{5}{3}$,

then we shall take LG = 5 units of length
and LH = 3 „ „


G
O
H
K
L
A
B
C
D
E
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P
Q
b
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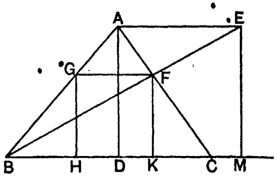
f

PROPOSITION 29. PROBLEM.

To construct a square such that two of its angular points lie on two sides of a given triangle, and one of its sides is in the third side of the triangle.



Let ABC be the $\triangle$, and draw AD perp^r to BC.

Draw AE perp^r and equal to AD, so that E lies on the opp. side of AC to B. Join EB cutting AC at F.

Draw FG $\parallel$ to CB cutting AB at G, and also draw GH and FK $\parallel$ to AD cutting BC at H and K.

GFKH *will be the sq. reqd.*

Draw EM perp^r to BC produced if necessary.

$$\frac{EM}{FK}=\frac{EB}{FB} \text{ (for the } \triangle\text{s EMB, FKB are similar)}$$

$$=\frac{AE}{GF} \text{ (,, BAE, BGF ,,).}$$

But EM = AD = AE.

$\therefore$ FK = FG

= HK = GH (for GFKH is a par^m).

Also the $\angle$GHK = the $\angle$ADC = a rt. $\angle$.

$\therefore$ GFKH is a square.

EXERCISES. PROBLEMS.

1. Divide a given straight line into five equal parts.

2. Divide a given straight line externally in the ratio of 5 to 3. LXIV. 2.

3. On a given straight line describe a rectangle equal to a given rectangle. LXIV. 3.

4. Describe a square equal to a given rectangle. LXIV. 4.

5. AB, ACD are two straight lines such that

AB : AC : CD = 3 : 4 : 6. LXIV. 5.

If the circum-circle of the triangle BCD cut AB again at E, find the ratio of BE to BA.

6. Construct a square whose area is 3·6 sq. in. Measure its side. LXIV. 6.

7. On a straight line 2·4 in. long, construct a rectangle whose area is 1 sq. in. LXIV. 7.

8. On a straight line 3·7 in. long, construct a rectangle whose area is equal to that of another rectangle whose adjacent sides are 1·7 and 2·9 inches. LXIV. 8.

9. Draw a triangle whose sides are 6, 6·3, and 8 cm. long. In it describe a square with one of its sides in the longest side of the triangle, and with angular points on the other two sides. LXIV. 9.

10. Describe a square equal in area to a given triangle. LXIV. 10.

11. Construct a square whose area is 2·8 sq. in. Measure its side. LXIV. 11.

12. Find by construction, as accurately as you can, the square root of 3·6. LXIV. 12.

13. Construct a triangle whose sides are 1·7, 2·6, and 3 in. Describe a square equal to it in area, and measure its side. LXIV. 13.

14. Construct a triangle similar to a given triangle and equal to one-ninth of its area. LXIV. 14.

15. Construct a triangle similar to a given triangle and equal to one-sixteenth of its area. LXIV. 15.

16. ABC is a given triangle. Construct a parallelogram four times its area, having AC for one side, and another side along AB. LXIV. 16.

17. From a given external point, draw a tangent to a given circle without using the centre. LXIV. 17.

18. Draw a rectangle having sides 3·12 inches and 1·28 inches. Construct a square equal to it. Measure and write down the length of one side and of one diagonal of the square (correct to two decimal places). State the construction. LXIV. 18.

19. Construct a triangle with two sides 4 and 5 inches long, and the contained angle 30°. Find another triangle of equal area but having two sides 3·25 and 4·5 inches respectively. LXIV. 19.

20. Construct a triangle with sides 2·4 in., 3·6 in., and 4·2 in. long: and construct a square equal to it in area. LXIV. 20.

21. A, B, C are three given straight lines. Find by construction a fourth straight line D such that $A^2 : B^2 = C : D$. LXIV. 21.

22. Employ Proposition V. 3 to trisect a given straight line. LXIV. 22.

23. Divide a given straight line in the ratio of 5 to 3. LXIV. 23.

24. Use Proposition V. 4 to divide a given straight line externally in the ratio of 3 to 2. LXIV. 24.

25. Trisect a given triangle by lines drawn parallel to its base. LXIV. 26.

26. Given the sum of the squares on two straight lines, and their ratio, find the straight lines. LXIV. 27.

27. Given the sum of the squares on three straight lines, and their ratios, find the straight lines. LXIV. 28.

28. Given the difference of the squares on two straight lines, and their ratio, find the straight lines. LXIV. 29.

29. Draw a triangle similar to a given triangle and equal to twice its area. LXIV. 30.

30. Draw a triangle similar to a given triangle and equal to three times its area. LXIV. 31.

31. Draw a triangle similar to a given triangle and equal to five times its area. LXIV. 32.

32. Construct an isosceles triangle equal to a given scalene triangle and having the same vertical angle. LXIV. 33.

33. From a given point O outside a given angle BAC draw a straight line cutting AB and AC at D and E so that OD is to DE in a given ratio. LXIV. 34.

34. Having given the vertical angle of a triangle, the ratio of the sides containing it, and the diameter of the circum-circle, describe the triangle. LXIV. 35.

35. Two straight lines AOC, BOD intersect in O and the lines AB, CD are drawn. From the greater of the two triangles AOB, COD cut off a part equal to the less by a straight line drawn through the point O. LXIV. 36.

36. Given that the base AB of a triangle is fixed, and 2·4 inches long, while the ratio of the sides AC and CB is 5 : 3, construct the locus of the vertex C. LXIV. 37.

37. Describe a rectangle so that its sides are in a given ratio, and its area is equal to a given square. LXIV. 38.

38. Construct two equilateral triangles which shall have a given ratio to each other and which shall be together equal to a given equilateral triangle. LXIV. 39.

39. Divide a straight line 3·8 in. long into two parts so that the rectangle contained by them is equal to a square of side 1·3 in. Measure the segments. LXIV. 40.

40. With ruler and set-square describe on a given straight line a figure similar to a given rectilineal figure of five sides ; the given straight line being parallel to one of the sides of the given figure. LXIV. 41.

41. Divide a triangle ABC into three equal areas, by lines drawn from a given point in one side. LXIV. 42.

42. Divide a given triangle into four equal areas by straight lines drawn through a given point within it. LXIV. 43.

43. Bisect a given triangle by a straight line drawn perpendicular to one side. LXIV. 44.

44. Bisect a given quadrilateral by a straight line drawn from a given point in one of its sides. LXIV. 45.

45. Trisect a given quadrilateral by straight lines drawn from a given point in one of its sides. LXIV. 46.

46. Trisect a given square by lines drawn parallel to one of its diagonals. LXIV. 47.

47. Construct a triangle having two sides in the ratio of 3 to 5, the included angle equal to 60°, and the third side 5 cm. long. LXIV. 48.

48. Construct a triangle having a vertical angle equal to 45°, the enclosing sides in the ratio of 3 to 4, and an altitude of 6·5 cm. LXIV. 49.

49. On a straight line 3·7 cm. long, describe a parallelogram having an area of 8·6 sq. cm., and one of its angles equal to 45°. LXIV. 50.

50. With the help of a protractor, construct a triangle, whose base AB = 2·7 in., one side AC = 3·1 in., and angle ACB (apex) = 41°. Also construct a triangle of equal area, and having the apex angle 63°, so that its base is in the same straight line AB, and one of its sides in BC. LXIV. 51.

51. Inscribe a square and an equilateral triangle in a circle of radius 1·73 inches. Find *by construction*, and write down, the ratio that the area of the square inscribed in the circle bears to the area of the square described on one of the sides of the equilateral triangle. LXIV. 52.

52. Through a given point draw a straight line to meet two others, such that the given point will divide it into two parts, of which one is double of the other. LXIV. 53.

53. Construct an equilateral triangle equal to a given square. LXIV. 54.

54. Construct a regular hexagon equal to a given square. LXIV. 55.

55. Construct a regular hexagon equal to a given equilateral triangle. LXIV. 56.

56. Through a given point within a circle draw a chord whose segments are in a given ratio. LXIV. 57.

57. Construct a triangle having given the base, the vertical angle, and the rectangle contained by the sides. LXIV. 58.

58. A triangle has two sides 2·3 and 3 in. long. Find by construction, what the length of the third side must be in order that the triangle may be equal to a square of 1 in. side. LXIV. 59.

59. Construct a rectangle whose area is 4·7 sq. in., and whose sides are in the ratio 3 : 1. LXIV. 60.

EXERCISES.

Loci.

1. From a fixed point O any straight line OP is drawn to a fixed straight line PA. In OP a point Q is taken so that OQ is to OP in a constant ratio : find the locus of Q. LXVI. 1.

[Draw OM perp^r to AP, and through Q draw QN perp^r to OM.

From the equiangular $\triangle$s OPM, OQN, $\frac{ON}{OM}=\frac{OQ}{OP}$ = a constant quantity;

$\therefore$ since M is a fixed pt., N is also a fixed pt.,

i.e. the locus of Q is a str. line || to the fixed line PA.]

2. From a fixed point O a straight line is drawn to meet a fixed circle in P ; in OP a point Q is taken so that OP is to OQ in a fixed ratio ; find the locus of Q. LXVI. 2.

[Take C the centre of the fixed circle, and in OC take a pt. D so that $\frac{OC}{OD}$ = the given ratio. $\frac{OC}{OD}=\frac{OP}{OQ}$; $\therefore$ DQ is || to CP and the $\triangle$s OCP, ODQ are similar; $\therefore$ $\frac{DQ}{CP}=\frac{OD}{OC}$, which is a constant ratio; $\therefore$ DQ is of constant length, and the locus of Q is a circle of centre D.]

3. Find the locus of a point which moves so that its distances from two fixed points are in constant ratio. LXVI. 3.

(Let A, B be the fixed pts. and P one position of the moving pt. Let the internal and external bisectors of the $\angle$P meet AB and AB produced at C and D. By V. 3 and 4 the locus will be seen to be a circle on CD as diameter.)

4. From a fixed point O straight lines are drawn to meet a given straight line: find the locus of points dividing these lines in the ratio of 2 to 1. LXVI. 4.

5. Find the locus of the middle points of the portions of straight lines intercepted between two given parallel straight lines. LXVI. 5.

6. Find the locus of the intersection of the medians of a triangle on a given base if the vertex lies on a given straight line. LXVI. 6.

7. Find the locus of the intersection of the medians of a triangle on a given base, and having a given vertical angle. LXVI. 7.

8. Find the locus of the intersection of the medians of a triangle on a given base, and having a given area. LXVI. 8.

9. A point moves so that its perpendicular distances from two given straight lines are in a constant ratio: find its locus. LXVI. 9.

10. A, B, C, D are given points in a straight line: find the locus of a point at which AB and CD subtend equal angles. LXVI. 10.

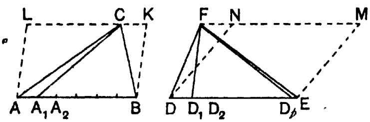
11. Find a point such that the perpendiculars from it upon the sides of a given triangle are proportional to three given straight lines. LXVI. 11.

12. A triangle whose angles are given has one angular point fixed, and another angular point moves on a given fixed straight line: prove that the locus of the third angular point is a straight line. LXVI. 12.

Proposition A. Theorem.

(For Incommensurables.)

Triangles of equal altitudes are to one another as their bases.



Let ABC, DEF be two $\triangle$s of equal altitudes on incommensurable bases AB, DE.

It is reqd. to prove that $\dfrac{\triangle ABC}{\triangle DEF} = \dfrac{AB}{DE}$.

Let AB be divided into any number (n) of equal parts AA_1, A_1A_2, ... $A_{n-1}B$.

Along DE set off parts DD_1, D_1D_2, ... each equal to AA_1, and let D_p be the nearest pt. of division to E.

$\triangle ACA_1 = \triangle DFD_1$ (on equal bases and of the same altitude). (II. 11.)

Also $\triangle ACB = n \,.\, \triangle ACA_1$, and $\triangle DFD_p = p \,.\, \triangle DFD_1 = p \triangle ACA_1$.

$$\therefore \frac{\triangle ACB}{\triangle DFD_p} = \frac{n}{p} = \frac{AB}{DD_p}.$$

By increasing n, the number of parts of AB_1, indefinitely, we can make the part D_pE as small as we please;

i.e. we can make DD_p and $\triangle DFD_p$ differ as little as we please from DE and $\triangle$ DFE respectively.

$\therefore$ making n infinitely large, we have ultimately,

$$\frac{\triangle ACB}{\triangle DFE} = \frac{AB}{DE}. \qquad \text{Q.E.D.}$$

COROLLARY. Since the par^{ms} ABKL, DEMN, on bases AB and DE and of the same altitude, are respectively double of the $\triangle$s ABC, DEF,

$$\therefore \frac{\text{par}^{m} \text{ AK}}{\text{par}^{m} \text{ DM}} = \frac{AB}{DE}.$$

EUCLID'S DEFINITION OF PROPORTION FOR INCOMMENSURABLE MAGNITUDES.

Four magnitudes are said to be proportional when, any equimultiples whatever of the first and third being taken, and any equimultiples whatever of the second and fourth, the multiple of the first is greater than, equal to, or less than that of the second, according as the multiple of the third is greater than, equal to, or less than that of the fourth.

Thus if A, B, C, D are four magnitudes such that

$$mA \gtreqless nB \text{ according as } mC \gtreqless nD,$$

whatever integral values m and n may have, A, B, C, D are in proportion.

For if the ratios $\frac{A}{B}$ and $\frac{C}{D}$ were unequal (say $\frac{A}{B}$ the greater), it would be possible to find a fraction $\frac{n}{m}$ lying between them in value.

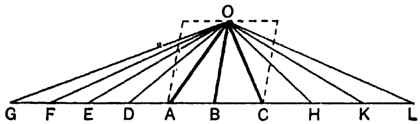
Then $$\frac{A}{B} > \frac{n}{m}, \text{ and } \frac{C}{D} < \frac{n}{m},$$

i.e. mA would be *greater* than nB while mC would be *less* than nD, contrary to the test stated above.

Proposition A. Theorem.

(Alternative Proof by means of Euclid's Def. of Proportion.)

Triangles and parallelograms of the same altitude are to one another as their bases.



Let the $\triangle$s OAB, OBC, and the par^{ms} OA, OC have the same altitude.

It is reqd. to prove that

$$(1)\quad \frac{\triangle OAB}{\triangle OBC}=\frac{AB}{BC};$$

$$(2)\quad \frac{\text{par}^m\, OA}{\text{par}^m\, OC}=\frac{AB}{BC}.$$

Produce AC both ways, and take any number of str. lines AD, DE, EF, FG each equal to AB; and any number CH, HK, KL each equal to BC.

Join OA, OD, etc., and OH, OK, etc.

(1) AB = AD = etc.; $\therefore$ $\triangle$AOB = $\triangle$OAD = etc. (II. 11, *Cor.* 2.)

$\therefore$ $\triangle$OGB is the same multiple of $\triangle$OAB that BG is of BA.

Similarly, $\triangle$OBL is the same multiple of $\triangle$OBC that BL is of BC.

Also, $\triangle OGB \gtreqless \triangle OBL$, according as $BG \gtreqless BL$.

Thus of the four magnitudes $\triangle$OAB, $\triangle$OBC, and the str. lines AB, BC, any equimultiples, viz. $\triangle$OGB and base GB, have been taken of the first and third, and any equimultiples, viz. $\triangle$OBL and base BL, have been taken of the second and fourth; and it has been shown that

$$\triangle OGB \gtreqless \triangle OBL, \text{ according as } BG \gtreqless BL.$$

$\therefore$ the four magnitudes are proportional;

$$i.e.\quad \frac{\triangle OAB}{\triangle OBC}=\frac{AB}{BC}.$$

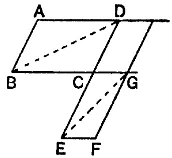
(2) Par^m OA = 2$\triangle$OAB, and par^m OC = 2$\triangle$OBC;

$$\therefore \frac{\text{par}^m\, AO}{\text{par}^m\, OC}=\frac{AB}{BC}.$$

Q.E.D.

PROPOSITION B. THEOREM.

Parallelograms which are equiangular to one another have to one another the ratio which is compounded of the ratios of their sides.



Let the par^m AC be equiangular to the par^m CF, having the $\angle$BCD $= \angle$ECG.

It is reqd. to prove that $\frac{par^m\ AC}{par^m\ CF}$ = *the ratio compounded of the ratios* $\frac{BC}{CG}$ *and* $\frac{DC}{CE}$.

Let the par^{ms} be placed so that BC and CG are in the same str. line.

Then DC and CE are in one str. line. (I. 2.)

Complete the par^m DG.

$$\frac{\text{par}^m\ AC}{\text{par}^m\ DG} = \frac{BC}{CG},$$

for the par^{ms} have equal altitudes; (V. A.)
and for the same reason

$$\frac{\text{par}^m\ DG}{\text{par}^m\ CF} = \frac{DC}{CE};$$

$\therefore$ the ratio compounded of $\frac{\text{par}^m\ AC}{\text{par}^m\ DG}$ and $\frac{\text{par}^m\ DG}{\text{par}^m\ CF}$

= the ratio compounded of $\frac{BC}{CG}$ and $\frac{DC}{CE}$,

i.e. by def. of compounded ratio,

$$\frac{\text{par}^m\ AC}{\text{par}^m\ CF} = \text{the ratio compounded of } \frac{BC}{CG} \text{ and } \frac{DC}{CE}.$$

Q.E.D.

COR. 1. This result may be stated thus:

$$\frac{\text{par}^m\ AC}{\text{par}^m\ CF} = \frac{\text{rect. } BC \cdot CD}{\text{rect. } CE \cdot CG},$$

i.e. parallelograms which are equiangular to one another are proportional to the rectangles contained by their adjacent sides.

COR. 2. *Triangles which have one angle of the one equal to one angle of the other are to one another in the ratio compounded of the ratios of the sides about the equal angles.*

Let BCD, ECG be two $\triangle$s, having their $\angle$s at C equal.

As they are halves of the par^{ms} AC, CF, *it follows that* $\frac{\triangle BCD}{\triangle ECG}$ = *the ratio compounded of* $\frac{BC}{CG}$ *and* $\frac{DC}{CE}$.

Q.E.D.

PROPOSITION C. THEOREM.

Equiangular parallelograms which are equal in area have their sides about the equal angles reciprocally proportional.

Conversely, equiangular parallelograms which have the sides about the equal angles reciprocally proportional, are equal in area.

Using the diagram of the previous Proposition,

(1) *We have to prove that* $\frac{BC}{CG} = \frac{CE}{CD}$.

$$\frac{BC}{CG} = \frac{\text{par}^m\ AC}{\text{par}^m\ DG}$$

$$= \frac{\text{par}^m\ CF}{\text{par}^m\ DG} \qquad \textit{Given.}$$

$$= \frac{EC}{CD}. \qquad \text{Q.E.D.}$$

(2) Conversely, if $\frac{BC}{CG} = \frac{CE}{CD}$,

We have to prove that par^m AC = par^m CF.

$$\frac{\text{Par}^m\ AC}{\text{Par}^m\ DG} = \frac{BC}{CG} = \frac{CE}{CD} \qquad \textit{Given.}$$

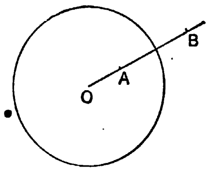
$$= \frac{\text{par}^m\ CF}{\text{par}^m\ DG};$$

$\therefore$ par^m AC = par^m CF. Q.E.D.

APPENDIX TO BOOK V

INVERSE POINTS.

DEFINITION.—If from O the centre of a circle of radius r a straight line OAB is drawn so that the rect. OA . OB $= r^2$, each of the points A and B is said to be the inverse of the other.

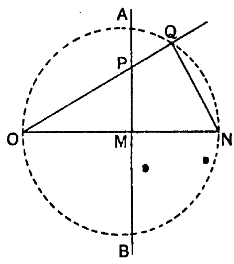


It is evident that, of the points A and B, one will be within the circle and the other without it.

If the pt. A moves on a given curve, it is evident that the pt. B will describe another curve.

The locus of A is called the inverse of the locus of B.

The inverse of a straight line.



It is reqd. to find the inverse of the str. line AB with respect to the fixed point O.

From O draw any str. line OPQ to meet AB in P so that OP . OQ $= r^2$, where r is constant.

Draw OM perp^r to AB and take a point N in it so that OM . ON $= r^2$.

OM . ON $= r^2 =$ OP . OQ ;

∴ P, Q, N, M are concyclic ;

∴ ∠PMN + ∠PQN = 2 rt. ∠s.

But ∠PMN is a rt. ∠, ∴ ∠PQN is a rt. ∠, ∴ the locus of Q is a circle on ON as diameter, for N is a fixed pt.

∴ the inverse of a str. line is a circle which passes through the fixed pt. O, and has its centre in the perp from O to the given line.

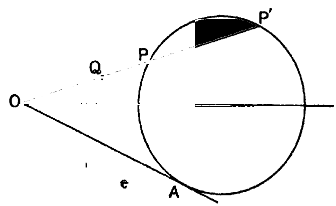
The inverse of a circle with respect to a point on its circumference.

From O a pt. on the given circle draw OQ to meet the circle in Q, and in OQ take a pt. P so that OP . OQ $= r^2$ (see diagram in the preceding problem).

Draw the diameter ON, and in it take a pt. M so that OM . ON $= r^2$.

As in the preceding problem, we can prove that $\angle$PMN is a rt. $\angle$, and thus we see that the locus of P is a str. line perp^{r} to ON.

The inverse of a circle with respect to a point not on its circumference.



Let O be the fixed pt., and from O draw OP to meet the given circle in P. Take a pt. Q in OP so that OP . OQ $= r^2$, where r is constant.

Let OP meet the circle again in P', and draw OA a tangent to the circle.

$$OP \,.\, OQ = r^2$$

and
$$OP \,.\, OP' = OA^2;$$

$$\therefore \frac{OQ}{OP'} = \frac{r^2}{OA^2}, \text{ a constant ratio.}$$

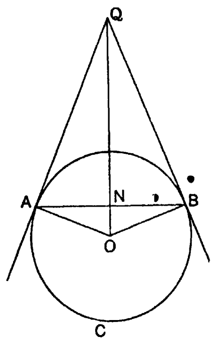
$\therefore$ from Ex. 2, p. 235, on Loci, we see that the locus of Q is a circle whose centre lies in the line from O to the centre of the given circle.

POLE AND POLAR.

Definition.—If tangents are drawn at the extremities of any chord of a circle which passes through a fixed point, the locus of their intersection is called the **polar** of the point. The fixed point is called the **Pole** of the Polar.

In connection with poles and polars of circles, the following proposition is most useful.

If tangents QA, QB *are drawn to a circle, the chord of contact* AB, *and* OQ, *the line joining* Q *to the centre, are at right angles.*



Let OQ meet AB at N. Join OA, OB.

In the $\triangle$s OAQ, OBQ,

(1) QA = QB,

(2) OA = OB,

(3) OQ is common;

$\therefore$ $\angle$AON = $\angle$BON.

Again in $\triangle$s AON, BON,

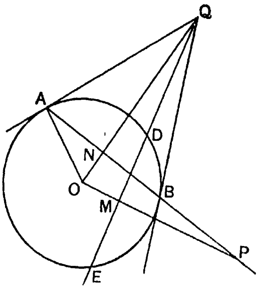
(1) OA = OB,

(2) ON is common,

(3) $\angle$AON = $\angle$BON;

$\therefore$ $\angle$ANO = $\angle$BNO, *i.e.* the $\angle$s at N are rt. $\angle$s. Q.E.D.

To find the polar of any fixed point P *with respect to a given circle.*



Let AB be any chord of the circle passing thro. the fixed pt. P; QA, QB, the tangents at A and B.

It is reqd. to find the locus of Q.

Take O the centre of the circle.

Join OQ meeting AB at N; and from Q draw QM perp^r to OP produced if necessary.

In the rt. angled $\triangle$OAQ, AN is perp^r to the hypotenuse, by the previous proposition;

$\therefore$ the $\triangle$s OAQ, ONA are similar; (V. 9.)

$$\therefore \frac{ON}{OA} = \frac{OA}{OQ};$$

$$\therefore ON \cdot OQ = OA^2.$$

The $\angle$s at M and N are rt. $\angle$s;

$\therefore$ the pts. Q, P, M, N are concyclic;

$$\therefore OP \cdot OM = ON \cdot OQ = OA^2.$$

But P is a fixed pt.; $\therefore$ M also is a fixed pt.

$\therefore$ the locus of Q is a str. line thro. the pt. M at rt. $\angle$s to OP.

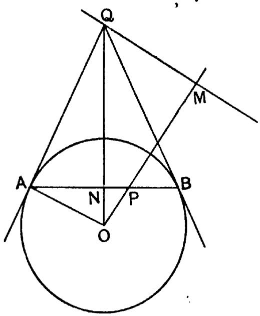
Hence the polar of a pt. P with respect to a circle of centre O is a str. line at rt. $\angle$s to OP, and cutting it at a pt. M so that OM . OP = the sq. on the rad. of the circle, *i.e.* at the pt. inverse to P.

COROLLARY 1. When the point P lies on the circumference of the circle, the polar of P is the tangent at P.

COROLLARY 2. When the point P lies without the circle, let QM meet the circle at D and E.

Then since $OM \cdot OP = OA^2 = OD^2 = OE^2$, we see that PD and PE are tangents.

$\therefore$ in this case, the polar of P is the chord of contact of the tangents drawn from P to the circle.



If the polar of P *passes through* Q, *the polar of* Q *passes through* P.

In the accompanying diagram, take Q any pt. on the polar QM of the pt. P, and draw PN perp^r to OQ. Let r be the radius of the circle. The $\angle$s at M and N are rt. $\angle$s; $\therefore$ the pts. Q, M, P, N are concyclic.

$$\therefore ON \cdot OQ = OP \cdot OM = r^2.$$

$\therefore$ PN is the polar of Q, *i.e.* the polar of Q passes thro. P.

Q.E.D.

EXERCISES.

ON POLES AND POLARS.

1. The straight line joining two points P and Q is the polar of the point of intersection of the polars of P and Q. LXVII. 1.

2. The point of intersection of two straight lines is the pole of the straight line joining their poles. LXVII. 2.

3. Find the locus of the poles of all straight lines which pass through a given point. LXVII. 3.

4. Any two points subtend at the centre of a circle an angle equal to that between their polars. LXVII. 4.

SALMON'S THEOREM.

5. The distances of any two points from the centre of a given circle are in the same ratio as the distance of each point from the polar of the other. LXVII. 5.

[Let P be the pole of MT, Q the pole of NT, O the centre; OM, QF perp^r to MT; ON, PE perp^r to NT; PR, QS perp^r to OQ, OP.

$$OS.OP = OQ.OR \text{ (for PSRQ is cyclic).}$$

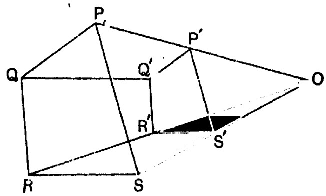
$$OP.OM = r^2 = OQ.ON;$$

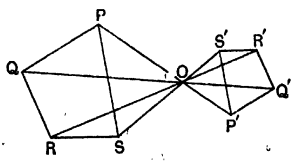
$$\therefore \frac{OM}{ON} = \frac{OQ}{OP} = \frac{OS}{OR} = \frac{OM-OS}{ON-OR} = \frac{SM}{RN};$$

$$\therefore \frac{OQ}{OP} = \frac{QF}{PE}. \text{]}$$

CENTRES OF SIMILARITY. SIMILITUDE.

If two unequal similar figures are placed so that their corresponding sides are parallel, the lines joining their corresponding points are concurrent.





Let PQRS, P′Q′R′S′ be two unequal similar and similarly situated figs., P, P′ being corresponding pts. and so on.

It is reqd. to prove that

PP′, QQ′, RR′, SS′ are concurrent.

Let PP′ and QQ′ meet in O.

PQ is ∥ to P′Q′; ∴ △s OPQ, OP′Q′ are similar.

$$\therefore \frac{OP}{OP'} = \frac{OQ}{OQ'} = \frac{PQ}{P'Q'},$$

i.e. PP′ and QQ′ are divided externally, or internally, at O in the ratio $\frac{PQ}{P'Q'}$.

In the same way QQ′ and RR′ meet at a pt. which divides them, externally or internally, in the ratio

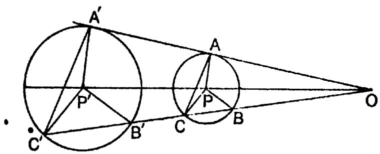
$$\frac{QR}{Q'R'}, \quad i.e. \text{ in the ratio } \frac{PQ}{P'Q'};$$

∴ PP′, QQ′, RR′ are concurrent.

Similarly SS′ passes thro. the same pt. O. Q.E.D.

Such a point O is called a **centre of similarity** of the two figures.

If a common tangent AA′ *to two circles and their line of centres* PP′ *meet at* O, *and from* O *a straight line be drawn to cut the circles at* B, C, *and* B′, C′ *respectively, the radii* PB, PC *are respectively parallel to the radii* P′B′, P′C′, *and the rectangles* OA . OA′, OB . OC′, OC . OB′ *are equal to one another.*



Join PB, PC, AC, P′B′, P′C′, A′C′.

∠PAO = a rt. ∠ = ∠P′A′O;

∴ △s PAO, P′A′O are similar;

$$\therefore \frac{OP}{OP'} = \frac{OA}{OA'} = \frac{PA}{P'A'}$$

$$= \frac{CP}{C'P'}.$$

∴ △s OCP, OC′P′ are similar;

∴ CP is ∥ to C′P′.

In the same way, PB is || to P′B′.

Also $\frac{OA}{OA'}=\frac{OP}{OP'}=\frac{OC}{OC'}$; $\therefore$ AC is || to A′C′.

Also rect. OC . OB $=OA^2$; $\therefore$ $\frac{OA}{OB}=\frac{OC}{OA}=\frac{OC'}{OA'}$;

$\therefore$ rect. OA . OA′ = rect. OB . OC′.

In the same way rect. OA . OA′ = rect. OC . OB′. Q.E.D.

Corollary. From the above we see that the external common tangents to two circles divide the line of centres externally in the ratio of the radii.

In the same way, it may be proved that the transverse common tangents divide the line of centres internally in the ratio of the radii.

Definition.—The points which divide the line of centres of two circles internally and externally in the ratio of the radii are called the **internal and external centres of similitude** of the circles.

EXERCISES.

On Centres of Similitude.

1. The straight line through the extremities of two parallel radii of two circles passes through one of their centres of similitude. LXVIII. 1.

2. The six centres of similitude of three circles taken in pairs lie three by three in four straight lines, called axes of similitude of the circles. LXVIII. 2.

3. If two circles touch two others, the radical axis of either pair passes through a centre of similitude of the other pair. LXVIII. 3.

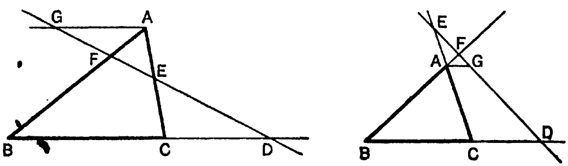
4. If a variable circle touch two fixed circles, the straight line joining the points of contact passes through a centre of similitude of the fixed circles. LXVIII. 4.

TRANSVERSALS.

Definition.—Any line, straight or curved, drawn to cut a system of other lines is called a transversal.

In the examples here given, straight lines only are dealt with.

If a transversal is drawn to cut the sides, or the sides produced, of a triangle, the product of three alternate segments is equal to the product of the other three segments.



Let ABC be a $\triangle$, and let a transversal cut BC, CA, AB, or those sides produced, at D, E, and F.

Then the product AE . BF . CD will be equal to the product AF . BD . CE. Through A draw AG parallel to BC to meet the transversal in G.

In the equiangular $\triangle$s BFD, AFG, $\dfrac{BF}{AF} = \dfrac{BD}{AG}$;

$\therefore$ the product AF . BD = the product BF . AG ....(1)

Also in the equiangular $\triangle$s CED, AEG, $\dfrac{CD}{AG} = \dfrac{CE}{AE}$;

$\therefore$ the product CD . AE = the product AG . CE;

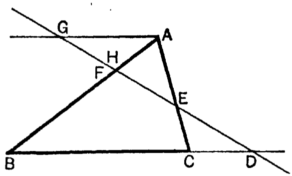
$\therefore$ from (1), $\dfrac{CD \cdot AE}{AF \cdot BD} = \dfrac{AG \cdot CE}{BF \cdot AG} = \dfrac{CE}{BF}$;

$\therefore$ AE . BF . CD = CE . AF . BD.

Q.E.D.

CONVERSE OF THE PRECEDING.

If two points are taken in the sides of a triangle, and a third point in the other side produced, or if three points are taken in the produced three sides of a triangle, such that the product of three alternate segments is equal to the product of the other three, the three points are collinear.



If possible let DE meet AB at H.

Then, by the preceding,

$$AH . BD . CE = BH . CD . AE.$$

But by hypothesis,

$$AF . BD . CE = BF . CD . AE;$$

$$\therefore \frac{AH}{AF} = \frac{BH}{BF},$$

i.e. AB is divided in the same ratio at F and H.

Also we see that the pts. F and H are both either within AB or in AB produced;

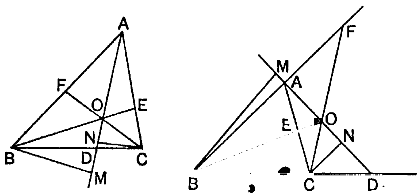
$\therefore$ the pts. F and H must be coincident,

i.e. D, E, F are collinear.

CEVA'S THEOREM.

If three concurrent straight lines are drawn from the vertices of a triangle to meet the opposite sides, produced if necessary, the product of three alternate segments, taken in order, is equal to the product of the other three.

Conversely, if three straight lines are drawn from the vertices of a triangle to cut the opposite sides (all the sides internally, or one internally and two externally) so that the product of three alternate segments, taken in order, is equal to the product of the other three, the three straight lines are concurrent.



Let ABC be a △, and let AD, BE, CF passing through a pt. O meet the opp. sides in D, E, F.

Then shall AF . BD . CE = BF . CD . AE.

Draw BM, CN perp[r] to AD.

In the equiangular △s BMD, CND,

$$\frac{BD}{CD} = \frac{BM}{CN} \text{..............................(1)}$$

$$= \frac{\triangle AOB}{\triangle AOC}$$

(for these △s are on the same base AO, and BM, CN are their altitudes).

Similarly $\qquad \dfrac{CE}{EA} = \dfrac{\triangle BOC}{\triangle AOB}$, ...(2)

and $\qquad \dfrac{AF}{BF} = \dfrac{\triangle AOC}{\triangle BOC}$; ....(3)

∵ from (1), (2) and (3), $\dfrac{AF . BD . CE}{BF . CD . AE} = 1$;

∴ AF . BD . CE = BF . CD . AE.

To prove the converse, let AD, BE cut at O, and if possible let CO produced cut AB at G.

By the above, AG . BD . CE = BG . CD . AE.

But by hypothesis AF . BD . CE = BF . CD . AE.

$$\therefore \frac{AG}{AF} = \frac{BG}{BF};$$

∴ as in the preceding theorem, the pts. F and G must be coincident.

i.e. AD, BE, CF are concurrent.

HARMONIC SECTION.

Definition.—When a finite straight line is divided internally and externally in the same ratio, it is said to be cut harmonically, and the four points thus obtained are said to be a Harmonic Range.

Thus if AB is divided internally at C, and externally at D, so that $\frac{AC}{CB} = \frac{AD}{D3}$, AB is cut harmonically at C and D; and the four points A, C, B, D form a harmonic range.

The points C and D are called Harmonic Conjugates, or are said to be harmonically conjugate to each other with respect to A and B.

If the four points A, C, B, D are joined to any point O outside the range, the four lines OA, OC, OB, OD are called a Harmonic Pencil. O is called the vertex of the Pencil, and any one of the four straight lines is called a Ray.

If a straight line AB *is divided internally and externally at* C *and* D *in the same ratio, then* CD *is divided internally and externally at* B *and* A *in the same ratio.*

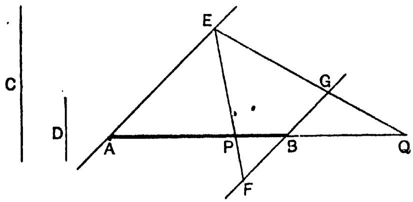
(See the preceding diagram.)

By hypothesis,
$$\frac{AC}{BC} = \frac{AD}{BD};$$

$$\therefore \frac{CA}{DA} = \frac{CB}{DB},$$

which proves the theorem.

To divide a given straight line AB *internally and externally in the same ratio as the two given straight lines* C *and* D.



Thro. A and B draw any two parallel str. lines AE, FBG; making AE = C and BF = BG = D, as shown in the diagram.

Let EF and EG produced meet AB in P and Q.

Then will $$\frac{AP}{PB} = \frac{AQ}{QB} = \frac{C}{D}.$$

By parallels, the $\triangle$s AEP, BFP are equiangular;

$$\therefore \frac{AP}{PB} = \frac{AE}{BF}$$

$$= \frac{C}{D}.$$

In the same way, $\triangle$s AEQ, BGQ are equiangular;

$$\therefore \frac{AQ}{BQ} = \frac{AE}{BG}$$

$$= \frac{C}{D}.$$

$\therefore$ the str. line AB is divided internally and externally at P and Q in the given ratio. Q.E.F.

If the four points A, P, B, Q *form a Harmonic range,* AQ, AB, AP *are in Harmonical Progression.*

By hypothesis, $$\frac{AQ}{BQ} = \frac{AP}{BP};$$

$\therefore$ alternately, $$\frac{AQ}{AP}=\frac{BQ}{BP}$$
$$=\frac{AQ-AB}{AB-AP};$$

$\therefore$ AQ, AB, AP are in Harmonical Progression.

COROLLARY. By a preceding proposition, PQ is divided internally and externally at B and A in the same ratio.

$\therefore$ the pts. Q, B, P, A form a Harmonic range.

$\therefore$ QA, QP, QB are also in Harmonical Progression.

EXERCISES.

ON HARMONIC SECTION.

1. Find the Harmonic Mean between two given straight lines.

(Take ABC so that AB and AC are equal to the given str. linesl On BC as diameter describe a circle, and from A draw tangents AD, AE to it. If the chd. of contact DE meets BC at F, AF will be the H.M. reqd. Prove that DB and DC bisect interior and exterior $\angle$s of the $\triangle$ADF, and hence, by V. 3 and 4, that AF is divided internally and externally in the same ratio at B and C.) LXIX. 1.

2. In the figure of the above example, if O is the centre of the circle, prove that AO is the Arithmetic and AD the Geometric Mean between AB and AC. Hence prove that the Arithmetic, the Geometric, and the Harmonic Means of two straight lines are in continued proportion. LXIX. 2.

3. Any line cutting a circle, and passing through a fixed point, is cut harmonically by the circle, the point, and the polar of the point. LXIX. 3.

4. If ABC be a triangle and CE a line through the vertex parallel to the base AB; then any transversal through D, the middle point of AB, will meet CE in a point which will be the harmonic conjugate of D, with respect to the points in which the transversal meets the sides of the triangle. LXIX. 4.

5. AB is divided harmonically at C and D, and a harmonic pencil, vertex O, is drawn. Through C a straight line ECF is drawn parallel to OD to meet OA, and OB at E and F. Prove that EC=CF. LXIX. 5.

Any transversal cutting a harmonic pencil is divided harmonically.
LXIX. 6.

7. Three points A, B, C being in the same straight line, find two points equidistant from C which divide the segment AB harmonically.
LXIX. 7.

8. All straight lines cutting the arms of an angle and the internal and external bisectors of the angle are divided harmonically.
LXIX. 8.